

Question 5

Let

$$X' = AX, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad X(0) = (0, 1)^T,$$

and introduce a time-dependent coordinate system $X = P(t)Z$, where

$$P(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

The physical state X and its coordinate column Z describe the same vector using different bases at different times.

Tasks

1. Derive the differential equation for Z by differentiating $X = PZ$. Explain why the constant-basis formula $P^{-1}AP$ is insufficient.
2. Compute the coefficient matrix and initial state for Z . Classify the transformed system as linear, homogeneous, and autonomous or nonautonomous.
3. Verify the supplied physical trajectory $X(t) = (\sin t, \cos t)^T$. Obtain $Z(t)$ and check both transformed equations directly.
4. Compare $X'(0)$ with $Z'(0)$. Explain how a zero instantaneous coordinate derivative can coexist with a changing physical vector.

Question 5 – Solution

Strategy. A moving basis contributes its own derivative; keep that contribution before multiplying by the inverse.

Step 1: Differentiate the whole change of coordinates. Since $X' = P'Z + PZ' = APZ$, invertibility of P gives

$$Z' = P^{-1}(AP - P')Z.$$

Dropping P' would treat a moving basis as fixed and generally produce incorrect coordinate derivatives. Here $\det P = 1$ for all real t .

Step 2: Compute the transformed system. The matrices are

$$P^{-1} = \begin{pmatrix} 1 & -t \\ 0 & 1 \end{pmatrix}, \quad AP - P' = \begin{pmatrix} 0 & 0 \\ -1 & -t \end{pmatrix}.$$

Consequently

$$Z' = \begin{pmatrix} t & t^2 \\ -1 & -t \end{pmatrix} Z, \quad Z(0) = (0, 1)^T.$$

This system is linear and homogeneous, but nonautonomous. Explicit time dependence can arise entirely from a time-dependent choice of coordinates.

Step 3: Check the supplied trajectory in both descriptions. For $X = (\sin t, \cos t)^T$, $X' = (\cos t, -\sin t)^T = AX$, with the stated data. Multiplication by P^{-1} gives

$$Z = (\sin t - t \cos t, \cos t)^T.$$

Its derivative is $(t \sin t, -\sin t)^T$. The transformed right-hand side is also $(t(\sin t - t \cos t) + t^2 \cos t, -(\sin t - t \cos t) - t \cos t)^T$, which simplifies to the same vector.

Step 4: Interpret the instantaneous difference. At 0, $X' = (1, 0)^T$ but $Z' = (0, 0)^T$. The identity $X' = P'Z + PZ'$ explains this: $P'(0)Z(0) = (1, 0)^T$. The basis motion accounts for the entire instantaneous change. The figure compares the first components; the coordinate curve starts with a horizontal tangent while the physical component has slope 1.

