

Question 4

Consider the first-order system on the real time axis

$$x' = x + y, \quad y' = 2x - y + t, \quad x(0) = 1, \quad y(0) = 2.$$

An equivalent scalar equation must preserve both the dynamics and the initial information; an eliminated coordinate must be recoverable.

Tasks

1. Eliminate y to derive a second-order equation for x . Find the two corresponding initial values.
2. Prove the converse: from any solution of your scalar IVP, reconstruct y and verify both original equations and both original initial values.
3. Instead eliminate x to obtain a scalar IVP for y and a reconstruction formula for x . Verify the forcing signs.
4. A calculation uses the correct scalar equation for x but imposes $x(0) = 1$, $x'(0) = 2$. Identify exactly which original initial condition this changes. Explain why prescribing both $x'(0) = 2$ and the original $y(0) = 2$ is inconsistent.

Question 4 – Solution

Strategy. Differentiate one row, substitute the other, and retain the algebraic formula that recovers the missing coordinate.

Step 1: Eliminate y while retaining initial information. The first row gives $y = x' - x$. Differentiating it and using the second gives $x'' = x' + 2x - y + t = 3x + t$. Thus

$$x'' - 3x = t, \quad x(0) = 1, \quad x'(0) = 3.$$

The initial slope is $x(0) + y(0)$, not the initial value of y alone.

Step 2: Reconstruct and verify. Given a twice differentiable solution of this scalar IVP, define $y = x' - x$. Then $x' = x + y$ identically, and

$$y' = x'' - x' = 3x + t - x' = 2x - (x' - x) + t = 2x - y + t.$$

Also $y(0) = 3 - 1 = 2$. This proves the converse as well as the forward implication: no extra scalar solutions remain after reconstruction and data matching.

Step 3: Eliminate in the other direction. The second row gives $x = (y' + y - t)/2$. Differentiating that row yields

$$y'' = 2x' - y' + 1 = 2x + 2y - y' + 1 = 3y + 1 - t.$$

Therefore the equivalent alternative is

$$y'' - 3y = 1 - t, \quad y(0) = 2, \quad y'(0) = 0, \quad x = \frac{y' + y - t}{2}.$$

Indeed $x' = (y'' + y' - 1)/2 = (3y + y' - t)/2 = x + y$; the reconstruction also gives $x(0) = 1$ and the second row directly.

Step 4: Diagnose the mismatched data. If $x(0) = 1$ and $x'(0) = 2$, reconstruction forces $y(0) = 2 - 1 = 1$. The calculation solves a different original IVP, with initial state $(1, 1)$. Insisting simultaneously on $y(0) = 2$ would require $x'(0) = 1 + 2 = 3$, contradicting the imposed slope. Correct differential equations alone do not make two initial-value problems equivalent.