

Question 2

Two masses move horizontally without friction. Their displacements u, v are measured to the right from equilibrium. Mass 1 has mass 1, is attached to a fixed left wall by a spring of stiffness 2, and is coupled to mass 2 by a spring of stiffness 3. Mass 2 has mass 2 and no wall spring. A force $f(t)$ acts to the right on mass 2. Use consistent mechanical units. Initially $(u, u', v, v') = (1, 0, 0, 0)$.

Tasks

1. Derive the two second-order force equations, checking the opposite signs of the coupling forces.
2. For $X = (u, u', v, v')^T$, give the first-order matrix system and initial state. Explain why (u, v) alone is insufficient.
3. When $f = 0$, differentiate $E = \frac{1}{2}(u')^2 + (v')^2 + u^2 + \frac{3}{2}(v - u)^2$ and prove conservation. For general f , find the exact power balance.
4. Find all equilibrium states for a constant force $f(t) = F$. Explain why equilibrium requires conditions on velocities as well as displacements.

Question 2 – Solution

Strategy. Apply Newton's law to each mass and use energy to check the coupling signs independently.

Step 1: Account for the spring forces. The wall spring exerts $-2u$ on mass 1. The coupling spring exerts $3(v - u)$ on mass 1 and its opposite on mass 2. Therefore

$$\boxed{u'' = -5u + 3v, \quad 2v'' = 3u - 3v + f(t).}$$

Dividing the second equation by its mass is essential.

Step 2: Include positions and velocities. Writing $p = u'$, $q = v'$ gives

$$X' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -5 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 3/2 & 0 & -3/2 & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ 0 \\ 0 \\ f(t)/2 \end{pmatrix}, \quad X(0) = (1, 0, 0, 0)^T.$$

Equal positions can have different velocities, so positions alone do not determine the first derivatives. The four coordinates supply the complete state.

Step 3: Verify the work-energy balance. Differentiate the stated energy before substituting the equations:

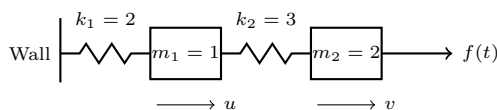
$$E' = p(-5u + 3v) + q(3u - 3v + f) + 2up + 3(v - u)(q - p).$$

Every spring term cancels, leaving $\boxed{E' = f(t)v'}$. Thus $f = 0$ conserves E ; for the stated initial data, $E(0) = 5/2$. The two kinetic terms reflect the different masses. The remaining terms are the nonnegative spring potential energies.

Step 4: Balance a constant force. An equilibrium has $p = q = 0$, $-5u + 3v = 0$, and $3u - 3v + F = 0$. Consequently $u = F/2$, $v = 5F/6$, and

$$\boxed{X_* = (F/2, 0, 5F/6, 0)^T.}$$

This is the unique equilibrium. Zero acceleration at one instant would not make a state constant if either velocity were nonzero.



Displacements from equilibrium; rightward positive.