

Question 10

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

The symmetric matrix

$$L = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

measures differences along the path 1-2-3. Let $\mathbf{1} = (1, 1, 1)^T$. A vector has zero mean when $x_1 + x_2 + x_3 = 0$.

Tasks

1. Find all eigenvalues and an orthogonal eigenvector basis. Verify each pair and show that the list is complete.
2. Express $x^T Lx$ as a sum of squares. Determine all vectors for which this expression is zero, and explain the zero eigenvalue.
3. Give a necessary and sufficient condition for $Lx = b$ to have a real solution. For $b = (2, -1, -1)^T$, find all solutions and the unique one with zero mean.
4. On the zero-mean subspace, prove the sharp bounds $\|x\|^2 \leq x^T Lx \leq 3\|x\|^2$ and give every equality case. Explain why the lower bound fails without the zero-mean restriction.

Question 10 – Solution

Strategy. Separate the constant direction, which the matrix cannot detect, from the two directions of variation.

Step 1: Verify a complete orthogonal basis. Direct multiplication gives

$$L\mathbf{1} = 0, \quad Lv = v, \quad v = (1, 0, -1)^T, \quad Lw = 3w, \quad w = (1, -2, 1)^T.$$

The three vectors are nonzero and mutually perpendicular, so they form a basis of $\mathbb{R}^3$. Thus the complete eigenvalue list is $[0, 1, 3]$, with the corresponding one-dimensional eigenspaces.

Step 2: Interpret the nonnegative quadratic form. Expanding gives

$$x^T Lx = (x_1 - x_2)^2 + (x_2 - x_3)^2 \geq 0.$$

It vanishes exactly when $x_1 = x_2 = x_3$, the constant direction. The zero eigenvalue reflects the fact that adding the same constant to all three entries changes no adjacent difference.

Step 3: Solve subject to the compatibility condition. Since $\mathbf{1}^T L = 0$, solvability requires $\mathbf{1}^T b = 0$. Conversely, every such b can be written $b = \alpha v + \beta w$, and $x = \alpha v + (\beta/3)w + c\mathbf{1}$ solves the system for every real c . For the stated b , dot products give $\alpha = 3/2$ and $\beta = 1/2$. Therefore all solutions are

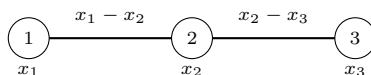
$$x = (5/3, -1/3, -4/3)^T + c\mathbf{1}, \quad c \in \mathbb{R}.$$

The displayed particular vector has zero mean; the total sum is $3c$, so zero mean uniquely selects $c = 0$.

Step 4: Prove the sharp restricted bounds. A zero-mean vector has form $x = av + bw$. Orthogonality and $\|v\|^2 = 2$, $\|w\|^2 = 6$ give

$$\|x\|^2 = 2a^2 + 6b^2, \quad x^T Lx = 2a^2 + 18b^2.$$

The lower equality holds exactly on the line through v ; the upper exactly on the line through w , with zero included in both. This proves both sharp bounds. A nonzero constant vector has positive length but zero quadratic form, so the lower bound cannot hold on all of $\mathbb{R}^3$.



Two edges of weight 1; energy is the sum of squared differences.