

Question 9

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = A^2.$$

For a polynomial p , define $p(A)$ by replacing each scalar power with the corresponding matrix power and each constant with that constant times I_3 .

Tasks

1. Find the eigenvalues and eigenspaces of A and B , and explain why squaring merges some eigenspaces.
2. Give a nonzero eigenvector of B that is not an eigenvector of A . Prove the general forward implication from an eigenpair of A to an eigenpair of $p(A)$, and explain why its converse can fail.
3. Find the monic polynomial of least degree satisfying $p(A) = 0$. Prove both the identity and the minimality of its degree.
4. Use that polynomial identity to express A^{-1} as a polynomial in A . Compute the inverse explicitly and verify it.

Question 9 – Solution

Strategy. A polynomial changes eigenvalues without changing each original eigenvector, but different eigenvalues can acquire the same image.

Step 1: Compare the two spectra. For A , the eigenspaces are

$$E_1 = \text{span}\{(1, 1, 0)^T\}, \quad E_{-1} = \text{span}\{(1, -1, 0)^T\}, \quad E_2 = \text{span}\{(0, 0, 1)^T\}.$$

These vectors form a basis. Since $B = \text{diag}(1, 1, 4)$, its eigenspace for 1 is the entire plane $x_3 = 0$, and its eigenspace for 4 is the third coordinate axis. Squaring sends both 1 and -1 to 1.

Step 2: Prove the forward rule and refute the converse. The vector $e_1 = (1, 0, 0)^T$ satisfies $Be_1 = e_1$ but $Ae_1 = (0, 1, 0)^T$, not a multiple of e_1 . In general, $Av = \lambda v$ implies $A^k v = \lambda^k v$ by induction; linearity then gives $p(A)v = p(\lambda)v$. The merged eigenspace permits new combinations that are eigenvectors of $p(A)$ without being eigenvectors of A .

Step 3: Find the least-degree annihilating polynomial. Any polynomial with $p(A) = 0$ must vanish at $1, -1, 2$, by applying the identity to the three nonzero eigenvectors. A nonzero polynomial therefore needs degree at least three. The monic candidate is

$$p(z) = (z - 1)(z + 1)(z - 2) = z^3 - 2z^2 - z + 2.$$

It annihilates all three basis eigenvectors, hence every vector, proving $p(A) = 0$ and minimality. It is the unique monic cubic with those roots.

Step 4: Recover the inverse from the identity. Rearrange $A^3 - 2A^2 - A + 2I = 0$ to get $A(-A^2 + 2A + I)/2 = I$. The factors commute, so the reverse product is also I . Thus

$$A^{-1} = \frac{-A^2 + 2A + I}{2} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}.$$

Direct multiplication confirms the inverse without division by a matrix expression.