

Question 8

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

For real τ , consider

$$A_\tau = \begin{pmatrix} 0 & 1 \\ -\tau & 2 \end{pmatrix}.$$

Investigate diagonalization over both $\mathbb{R}$ and $\mathbb{C}$.

Tasks

1. Find the characteristic polynomial and classify its roots for $\tau < 1$, $\tau = 1$ and $\tau > 1$. Check trace and determinant against the roots.
2. Find every eigenspace and classify diagonalizability in both fields, including the repeated-root case.
3. For $\tau > 1$, write a complex eigenvector as $p + iq$ and construct a real basis in which A_τ has a real 2×2 block representing its complex pair.
4. As $\tau \rightarrow 1^-$, examine a real eigenvector basis and its inverse. Explain why the limiting matrix can fail to be diagonalizable even though every matrix before the limit is diagonalizable.

Question 8 – Solution

Strategy. Follow the eigenvectors as well as the roots. Distinct roots can merge into one direction at the transition.

Step 1: Classify the roots. The polynomial is $\lambda^2 - 2\lambda + \tau$. For $\tau < 1$, set $d = \sqrt{1 - \tau} > 0$; the roots are $1 \pm d$. At $\tau = 1$ the root is 1 twice. For $\tau > 1$, set $b = \sqrt{\tau - 1} > 0$; the roots are $1 \pm ib$. Their sum is $2 = \text{tr } A_\tau$ and product is $\tau = \det A_\tau$ in all cases.

Step 2: Find the eigenspaces and field restrictions. The first row of $A_\tau x = \lambda x$ forces $x_2 = \lambda x_1$. The second row is then exactly the characteristic equation. Thus every root has complex eigenspace spanned by $(1, \lambda)^T$; for real roots, the real eigenspace is the real span of the same vector. At $\tau = 1$ this is only one-dimensional, despite algebraic multiplicity 2. Consequently

	$\mathbb{R}$ -diagonalizable	$\mathbb{C}$ -diagonalizable
$\tau < 1$	yes	yes
$\tau = 1$	no	no
$\tau > 1$	no	yes

The real failure above 1 comes from nonreal eigenvalues, not a shortage of complex eigenvectors.

Step 3: Construct a real representation for the complex pair. For $\lambda = 1 + ib$, write $(1, 1 + ib)^T = p + iq$ with $p = (1, 1)^T$, $q = (0, b)^T$. Equating parts gives $A_\tau p = p - bq$ and $A_\tau q = bp + q$. Since $\det(p \ q) = b \neq 0$, this is a real basis, with representation

$$(p \ q)^{-1} A_\tau (p \ q) = \begin{pmatrix} 1 & b \\ -b & 1 \end{pmatrix}.$$

It is a real block, not a real diagonalization.

Step 4: Inspect the collapsing eigenvector basis. Below 1, use $P = \begin{pmatrix} 1 & 1 \\ 1 + d & 1 - d \end{pmatrix}$. Its determinant is $-2d$, and

$$P^{-1} = \frac{1}{2d} \begin{pmatrix} d - 1 & 1 \\ 1 + d & -1 \end{pmatrix}.$$

In particular, the second column is unbounded as $d \rightarrow 0^+$. Both columns tend to $(1, 1)^T$ as $d \rightarrow 0^+$. The limiting basis is singular and cannot diagonalize the limiting matrix. Diagonalizability need not persist when distinct eigendirections coalesce.