

Question 7

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

For a real parameter K , let

$$T_K = \begin{pmatrix} 1/2 & K \\ 0 & 1/4 \end{pmatrix}.$$

Both eigenvalues have magnitude less than one. We compare long-term iteration with the stronger requirement $\|T_K x\| \leq \|x\|$ for every real vector x .

Tasks

1. Find an eigenvector basis for every K and determine whether the matrix is diagonalizable.
2. Derive T_K^m for every nonnegative integer m and prove that every fixed starting vector tends to zero under iteration.
3. For the starting vector $e_2 = (0, 1)^T$, find the first-step length. Determine exactly when that first step increases length, and explain why this does not contradict the long-term limit.
4. Find the necessary and sufficient condition on K for $\|T_K x\| \leq \|x\|$ to hold for every real x . Prove it by completing a square, rather than testing only e_2 .

Question 7 – Solution

Strategy. Decaying eigencomponents need not make Euclidean length decrease at every step when the eigenvectors are not orthogonal.

Step 1: Find the distinct-eigenvalue basis. The eigenvalues are $1/2$ and $1/4$, with eigenvectors $u = (1, 0)^T$ and $v = (-4K, 1)^T$. These are independent for every K , so T_K is always diagonalizable over $\mathbb{R}$.

Step 2: Compute all powers. For $m \geq 1$, the upper-right entry in the product is

$$K \sum_{j=0}^{m-1} (1/2)^{m-1-j} (1/4)^j = 4K(2^{-m} - 4^{-m}).$$

This finite geometric sum yields

$$T_K^m = \begin{pmatrix} 2^{-m} & 4K(2^{-m} - 4^{-m}) \\ 0 & 4^{-m} \end{pmatrix}.$$

The formula is I at $m = 0$. Every entry tends to zero for each fixed K , so $T_K^m x \rightarrow 0$ for every fixed starting vector x .

Step 3: Exhibit a transient length increase. The first image of e_2 is $(K, 1/4)^T$, of length $\sqrt{K^2 + 1/16}$. It exceeds 1 exactly when $|K| > \sqrt{15}/4$. An early increase and eventual decay concern different times, so there is no contradiction. The plot uses $K = 2$ and shows the actual lengths at integer iteration counts; the initial increase is followed by decay.

Step 4: Test all directions, not just one. For $x = (a, b)^T$, direct expansion and completion of the square give

$$\|T_K x\|^2 - \|x\|^2 = -\frac{3}{4} \left(a - \frac{2K}{3}b \right)^2 + \left(\frac{4K^2}{3} - \frac{15}{16} \right) b^2.$$

This is nonpositive for every a, b exactly when the last coefficient is nonpositive: necessity follows by choosing $a = 2Kb/3$ with $b \neq 0$. Hence the exact condition is

$$|K| \leq \frac{3\sqrt{5}}{8}.$$

This stricter threshold cannot be recovered by checking only e_2 or only the eigenvalues. At equality, nonzero vectors on $a = 2Kb/3$ retain their length in one step; outside that line they decrease.

