

Question 6

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} p & q \\ r & s \end{pmatrix}.$$

Two matrices commute when $AB = BA$.

Tasks

1. Find the full family of real matrices B commuting with A , by equating entries of the two products.
2. Find a common eigenvector basis for every matrix in this family. Explain why simultaneous diagonalization always implies commutation.
3. For $B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$, find the eigenvalues of AB and $A + B$ using the common basis. Explain how two invertible matrices can have a singular sum.
4. Within the commuting family, classify every invertible B for which $A + B$ is singular. Identify when the sum has a one-dimensional kernel and when it is the zero matrix.

Question 6 – Solution

Strategy. Commutation forces preservation of this matrix's two distinct eigendirections, allowing scalar tests in a common basis.

Step 1: Equate the products. Subtracting the products gives

$$AB - BA = \begin{pmatrix} r - q & s - p \\ p - s & q - r \end{pmatrix}.$$

Thus $r = q$ and $s = p$, with no further restrictions. Thus the entire commuting family is $B = \begin{pmatrix} p & q \\ q & p \end{pmatrix}$.

Step 2: Use the common directions. For $u = (1, 1)^T$ and $v = (1, -1)^T$, the eigenvalues of A are 3, 1, and those of B are $p + q, p - q$, respectively. These independent vectors form a common eigenvector basis, even if the two eigenvalues of B coincide. Conversely, if $A = PDP^{-1}$ and $B = PEP^{-1}$ with D, E diagonal, then $AB = PDEP^{-1} = PEDP^{-1} = BA$.

Step 3: Analyze the concrete sum and product. For $p = 1, q = 2$, the eigenvalues of B on u, v are 3, -1 . Therefore the eigenvalues of AB are 9, -1 and those of $A + B$ are 6, 0. Both A and B are invertible, but their actions on v cancel in the sum. The sum has kernel $\text{span}\{v\}$.

Step 4: Classify all cancellation cases. Invertibility of B requires $(p + q)(p - q) \neq 0$. Singularity of the sum requires $(p + q + 3)(p - q + 1) = 0$. Equivalently the full allowed set is

$$\{p + q = -3, p - q \neq 0\} \cup \{p - q = -1, p + q \neq 0\}.$$

On the first line only, the kernel is the line through u ; on the second only, it is the line through v . Their intersection is $(p, q) = (-2, -1)$, where $B = -A$ and $A + B = 0$, so the kernel is the entire plane. The excluded points are $(-3/2, -3/2)$ and $(-1/2, 1/2)$, where B itself is singular. The diagram marks these exclusions with open circles.

