

Question 5

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

A real 2×2 matrix A is required to have

$$Au = 2u, \quad Av = -v, \quad u = (1, 1)^T, \quad v = (2, -1)^T.$$

Tasks

1. Construct the unique matrix satisfying these specifications and verify both eigenpairs.
2. Explain the effect of rescaling the two eigenvectors or swapping their order together with the corresponding eigenvalues. Contrast this with assigning the eigenvalues to the opposite directions.
3. Can the same matrix also have $(1, 0)^T$ as an eigenvector for some eigenvalue? Prove your answer.
4. Keeping the two prescribed directions, replace their eigenvalues by arbitrary real r, s . Find the full matrix formula and explain how its eigenspaces change when $r = s$.

Question 5 – Solution

Strategy. Independent eigenvectors form an input basis, so their prescribed images determine every column of the transformation.

Step 1: Build the spectral representation. The basis matrix and inverse are

$$P = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad P^{-1} = \frac{1}{3} \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}.$$

Therefore

$$A = P \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} P^{-1} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}.$$

Its products with u, v are $(2, 2)^T = 2u$ and $(-2, 1)^T = -v$. Since u, v span the plane, these images determine A uniquely.

Step 2: Separate notation changes from changed assignments. Multiplying either basis column by a nonzero scalar leaves its eigenvalue relation unchanged, so the reconstructed matrix stays the same. Reordering columns and the matching diagonal entries also leaves the map unchanged. Assigning -1 to u and 2 to v instead produces $I - A = \begin{pmatrix} 1 & -2 \\ -1 & 0 \end{pmatrix}$, a different map. The correspondence between each direction and its scalar matters.

Step 3: Test a proposed third direction. Directly, $A(1, 0)^T = (0, 1)^T$, not a multiple of $(1, 0)^T$. Thus no choice of eigenvalue makes the additional request possible. Equivalently, $(1, 0)^T = (u + v)/3$ combines both distinct eigendirections.

Step 4: Recover the general two-scalar family. The same basis computation yields

$$A_{r,s} = \frac{1}{3} \begin{pmatrix} r+2s & 2r-2s \\ r-s & 2r+s \end{pmatrix}.$$

For $r \neq s$, the eigenspaces are exactly the two specified lines. For $r = s$, the matrix becomes rI_2 and every nonzero vector is an eigenvector; the eigenspace is the whole plane. Repeating an eigenvalue here enlarges the eigenspace, unlike a defective repeated-root example.