

Question 4

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Consider the real symmetric matrix

$$S = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Tasks

1. Find an orthonormal eigenvector basis and an orthogonal diagonalization $S = QDQ^T$. Verify the columns directly.
2. Prove that eigenvectors of a real symmetric matrix belonging to distinct real eigenvalues are perpendicular, and connect the proof to your basis.
3. Find the sharp minimum and maximum of $x^T S x$ over real vectors with $\|x\| = 1$, giving every equality case.
4. Use the spectral decomposition to find S^{-1} and verify it. Explain why a negative eigenvalue does not prevent invertibility but does prevent positive definiteness.

Question 4 – Solution

Strategy. An orthonormal eigenvector basis turns both a quadratic form and an inverse into scalar calculations.

Step 1: Construct and check the basis. Take

$$q_1 = \frac{1}{\sqrt{2}}(1, 1, 0)^T, \quad q_2 = \frac{1}{\sqrt{2}}(1, -1, 0)^T, \quad q_3 = (0, 0, 1)^T.$$

Direct multiplication gives eigenvalues 3, 1, -1, respectively. The columns are orthonormal, so $Q = (q_1 \ q_2 \ q_3)$ has $Q^T Q = I$. Thus $SQ = QD$ with $D = \text{diag}(3, 1, -1)$, and $S = QDQ^T$.

Step 2: Prove perpendicularity. For a real symmetric matrix, if $Sa = \lambda a$ and $Sb = \mu b$, then

$$\lambda a^T b = (Sa)^T b = a^T S^T b = a^T S b = \mu a^T b.$$

When $\lambda \neq \mu$, this forces $a^T b = 0$. Normalizing the three perpendicular directions gives exactly the orthonormal basis above.

Step 3: Derive the sharp quadratic-form bounds. Write $x = c_1 q_1 + c_2 q_2 + c_3 q_3$. For a unit vector, $c_1^2 + c_2^2 + c_3^2 = 1$, and

$$x^T S x = 3c_1^2 + c_2^2 - c_3^2.$$

This is a weighted average of 3, 1, -1, so $-1 \leq x^T S x \leq 3$. The minimum occurs exactly at $x = \pm q_3$; the maximum exactly at $x = \pm q_1$. The coefficients of all other directions must vanish at an endpoint.

Step 4: Invert the nonzero eigenvalues. Since no eigenvalue is zero,

$$S^{-1} = Q \text{diag}(1/3, 1, -1) Q^T = \begin{pmatrix} 2/3 & -1/3 & 0 \\ -1/3 & 2/3 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Multiplying by S gives I_3 . Invertibility excludes zero eigenvalues, not negative ones. But $q_3^T S q_3 = -1 < 0$, so S is not positive definite; in fact its quadratic form takes both positive and negative values.