

Question 3

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Allow complex eigenvalues and eigenvectors for the real matrix

$$C = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$

Write $i^2 = -1$, and let $e_1 = (1, 0)^T$.

Tasks

1. Find both complex eigenvalues and their eigenspaces, and verify a representative eigenvector for each directly.
2. Prove that there is no nonzero real eigenvector, despite the existence of a complex eigenvector basis.
3. Write an eigenvector for $1 + 2i$ as $p + iq$ with p, q real, and derive the two real equations obtained from its eigenvalue equation.
4. Express C as a positive scale times a real rotation. Find $C^m e_1$ and its length for every nonnegative integer m , with the rotation angle specified unambiguously.

Question 3 – Solution

Strategy. Complex eigenvectors encode a real invariant plane; they need not describe a real line fixed in direction.

Step 1: Find and verify the complex pairs. The polynomial is $(\lambda - 1)^2 + 4$, so

$$\lambda_{\pm} = 1 \pm 2i, \quad E_{1+2i} = \text{span}_{\mathbb{C}}\{(1, -i)^T\}, \quad E_{1-2i} = \text{span}_{\mathbb{C}}\{(1, i)^T\}.$$

Indeed $C(1, -i)^T = (1 + 2i, 2 - i)^T = (1 + 2i)(1, -i)^T$. Conjugation verifies the other pair. The distinct eigenvalues give independent complex eigenvectors.

Step 2: Exclude a real eigenvector. If a real nonzero x satisfied $Cx = \lambda x$, a nonzero coordinate of x would make λ the ratio of two real numbers. Thus λ would be real, but the characteristic polynomial has no real root. No such vector exists.

Step 3: Extract the real plane relations. For $(1, -i)^T = p + iq$, take $p = (1, 0)^T$, $q = (0, -1)^T$. Equating real and imaginary parts of $C(p + iq) = (1 + 2i)(p + iq)$ gives

$$Cp = p - 2q, \quad Cq = 2p + q.$$

Direct multiplication verifies both. These independent real vectors span the plane even though neither is a real eigenvector.

Step 4: Recover the rotation and scale. Let $\theta \in (0, \pi/2)$ satisfy $\cos \theta = 1/\sqrt{5}$ and $\sin \theta = 2/\sqrt{5}$; equivalently $\theta = \arctan(2)$. Then

$$C = \sqrt{5} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad C^m e_1 = 5^{m/2} (\cos m\theta, \sin m\theta)^T.$$

Its length is $5^{m/2}$. The figure shows e_1 and its image $(1, 2)^T$, with unit and image circles drawn on equal scales.

