

Question 2

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

For real a , consider

$$B_a = \begin{pmatrix} 2 & a \\ 0 & 2 \end{pmatrix}.$$

A student claims that a repeated eigenvalue automatically provides two independent eigenvectors.

Tasks

1. Find the eigenvalue and both multiplicities for every a , treating $a = 0$ separately.
2. Classify diagonalizability over $\mathbb{R}$ and over $\mathbb{C}$. Explain whether allowing complex eigenvectors repairs any failure.
3. Find an exact formula for B_a^m for every nonnegative integer m . Derive it rather than assuming diagonalization.
4. For a fixed vector $x = (x_1, x_2)^T$, classify when $2^{-m} B_a^m x$ stays bounded as $m \rightarrow \infty$. Compare this result with what the repeated eigenvalue alone suggests.

Question 2 – Solution

Strategy. Root multiplicity and the number of independent eigenvectors answer different questions.

Step 1: Solve the eigenspace equation. The characteristic polynomial is $(\lambda - 2)^2$ for every a . The equation $(B_a - 2I)x = 0$ reduces to $ax_2 = 0$. Thus the algebraic multiplicity is always 2, while

$$a = 0 : E_2 = \mathbb{R}^2, \dim E_2 = 2; \quad a \neq 0 : E_2 = \text{span}\{(1, 0)^T\}, \dim E_2 = 1.$$

A repeated root does not itself produce a second independent vector.

Step 2: Classify diagonalization in both fields. At $a = 0$, $B_a = 2I$ is already diagonal. At $a \neq 0$, there is only one independent eigenvector, so no eigenvector basis exists. The same equation $ax_2 = 0$ holds over $\mathbb{C}$, with the same dimension count there. Hence B_a is diagonalizable exactly when $a = 0$ over either field.

Step 3: Derive the powers from a terminating product. Let $N = B_a - 2I = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$, so $N^2 = 0$. The commuting binomial expansion gives

$$B_a^m = 2^m \begin{pmatrix} 1 & ma/2 \\ 0 & 1 \end{pmatrix}.$$

For $m = 0$ this is I . Multiplication by B_a adds $a/2$ to the scaled upper-right entry, confirming the formula inductively without an eigenvector basis.

Step 4: Identify the remaining growth. The scaled iterate is

$$2^{-m} B_a^m x = (x_1 + max_2/2, x_2)^T.$$

It is bounded exactly when $ax_2 = 0$, and then is constant. For every $a \neq 0$ and $x_2 \neq 0$, its first coordinate grows in magnitude linearly. Dividing out the common exponential factor 2^m does not remove this extra growth. The missing eigenvector permits behavior that the list of eigenvalues, even with multiplicity, does not describe by itself.