

Question 1

An eigenpair satisfies $Av = \lambda v$ with $v \neq 0$. The eigenspace $E_\lambda = \ker(A - \lambda I)$ includes zero, although zero is not an eigenvector. Algebraic multiplicity counts roots of $\det(\lambda I - A)$; geometric multiplicity is $\dim E_\lambda$. Work over $\mathbb{R}$ unless complex scalars are explicitly requested.

Let

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \quad u = (1, 1)^T, \quad v = (1, -1)^T,$$

and write a real starting vector as $x_0 = \alpha u + \beta v$.

Tasks

1. Find the eigenvalues, their full eigenspaces and both multiplicities. Explain why arbitrary nonzero multiples of an eigenvector are allowed but the zero vector is not.
2. Express $w = (3, 1)^T$ in the eigenvector basis and decide whether w itself is an eigenvector.
3. Find $A^m x_0$ for every nonnegative integer m , and specialize to w . Verify the result without multiplying out m matrices.
4. For every nonzero real x_0 , determine the limit of $A^m x_0 / \|A^m x_0\|$. Include the cases $\alpha > 0$, $\alpha < 0$ and $\alpha = 0$.

Question 1 – Solution

Strategy. Resolve the starting vector into invariant directions before analyzing repeated multiplication.

Step 1: Compute the eigenspaces. The characteristic polynomial is $(\lambda - 2)^2 - 1 = (\lambda - 3)(\lambda - 1)$. Solving the two singular systems gives

$$E_3 = \text{span}\{u\}, \quad E_1 = \text{span}\{v\}.$$

Both algebraic and geometric multiplicities are 1. Scaling a nonzero eigenvector preserves its eigenvalue equation. Zero solves every homogeneous eigenvalue equation and supplies no direction, so it is excluded as an eigenvector.

Step 2: Decompose and test the given vector. The equations $\alpha + \beta = 3$, $\alpha - \beta = 1$ give $w = 2u + v$. Both components are nonzero and their eigenvalues differ. Directly, $Aw = (7, 5)^T$ is not a scalar multiple of $(3, 1)^T$, so w is not an eigenvector.

Step 3: Compute all iterates. Linearity and $Au = 3u$, $Av = v$ give, by induction,

$$A^m x_0 = 3^m \alpha u + \beta v, \quad A^m w = (2 \cdot 3^m + 1, 2 \cdot 3^m - 1)^T.$$

The formula holds at $m = 0$, and one multiplication advances each coefficient by its eigenvalue, proving the next case.

Step 4: Classify normalized direction limits. If $\alpha \neq 0$, divide the iterate by $3^m |\alpha|$ before normalizing. The remaining v coefficient tends to zero, so the limit is $\boxed{u/\sqrt{2}}$ for $\alpha > 0$ and $\boxed{-u/\sqrt{2}}$ for $\alpha < 0$. If $\alpha = 0$, then $\beta \neq 0$ and the iterate is the unchanged vector βv , giving limit $\boxed{\text{sgn}(\beta)v/\sqrt{2}}$. The dominant eigenvalue determines the limit only when its component is present. The diagram shows the two directions and the decomposition $w = 2u + v$.

