

**Question 10**

Work with real matrices and column vectors. Write  $I_n$  for the  $n \times n$  identity,  $A^T$  for transpose, and  $\|v\| = \sqrt{v^T v}$  for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Let

$$N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A = I_3 + N.$$

A discrete update is  $v_{m+1} = Av_m$ , starting from  $v_0 = (0, 0, 1)^T$ . The index  $m$  is a nonnegative integer, and each update uses the entire old vector.

**Tasks**

1. Compute  $N^2, N^3$ ,  $\det A$  and  $A^{-1}$ . Verify the inverse without a general inverse algorithm.
2. Prove a finite formula for  $A^m$  for all nonnegative integers  $m$ , using induction or the binomial identity with justification.
3. Find  $v_m$  and describe its coordinate growth. Explain why  $\det A = 1$  does not imply bounded iterates.
4. A program first updates  $y \leftarrow y + z$  and then  $x \leftarrow x + y$ , using the already updated  $y$ . Find the matrix it actually applies and its trajectory from the same initial vector. Compare the two after two steps.

## Question 10 – Solution

**Strategy.** The powers of the strictly upper triangular part terminate. A matrix update is simultaneous unless a different order is explicitly encoded.

**Step 1: Exploit the terminating powers.** Multiplication gives  $N^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$  and  $N^3 = 0$ . The upper triangular matrix  $A$  has determinant 1. Moreover

$$A^{-1} = I_3 - N + N^2$$

because  $(I_3 + N)(I_3 - N + N^2) = I_3 + N^3 = I_3$ ; the reverse product is the same.

**Step 2: Derive the finite power formula.** The identity matrix commutes with  $N$ , so the binomial expansion terminates:

$$A^m = I_3 + mN + \frac{m(m-1)}{2}N^2.$$

Alternatively, the formula is true at  $m = 0$ . Multiplication by  $I_3 + N$  increases the coefficient of  $N$  to  $m + 1$  and that of  $N^2$  to  $m(m-1)/2 + m = m(m+1)/2$ , establishing the induction step.

**Step 3: Find the exact iterates.** Applying the power formula to  $v_0$  gives

$$v_m = (m(m-1)/2, m, 1)^T.$$

The first coordinate grows quadratically, the second linearly and the third stays fixed. Determinant 1 preserves oriented volumes of three-dimensional sets, not individual lengths or boundedness under repetition. The plot marks integer-time values only; coincident coordinate values share a location.

**Step 4: Diagnose the sequential-update error.** The program computes  $y_{\text{new}} = y + z$  and then  $x_{\text{new}} = x + y + z$ . It actually applies

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = I_3 + N + N^2.$$

Starting from  $v_0$ , its third coordinate remains 1, its second becomes  $m$ , and its first accumulates  $1 + 2 + \dots + m = m(m+1)/2$ . After two steps the correct vector is  $(1, 2, 1)^T$ , while the program gives  $(3, 2, 1)^T$ . Both matrices are invertible with determinant 1, so checking only the determinant would not expose this order-of-update mistake.

