

**Question 7**

Work with real matrices and column vectors. Write  $I_n$  for the  $n \times n$  identity,  $A^T$  for transpose, and  $\|v\| = \sqrt{v^T v}$  for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

For real  $t$ , let

$$M(t) = \begin{pmatrix} 1 & t \\ 0 & 2 \end{pmatrix}, \quad v(t) = \begin{pmatrix} t^2 \\ e^t \end{pmatrix}, \quad y(t) = M(t)v(t).$$

Differentiate matrices entry by entry.

**Tasks**

1. Derive the matrix-vector product rule from component sums. Compute  $y$  and  $y'$  in this example by two routes.
2. Find  $N(t) = M(t)^{-1}$ . Derive the inverse differentiation rule from  $MN = I_2$  and verify it here.
3. Compute  $(M^2)'$ ,  $M'M + MM'$  and  $2MM'$ . Decide whether the scalar shortcut for differentiating a square is valid.
4. Compute  $\det M(t)$  and examine  $\|M(t)(0, 1)^T\|$ . Explain why a constant nonzero determinant does not force a constant matrix or constant vector lengths.

## Question 7 – Solution

**Strategy.** Scalar differentiation still applies to each entry, but matrix factors must retain their multiplication order.

**Step 1: Derive and apply the product rule.** For component  $i$ ,  $y_i = \sum_j M_{ij}v_j$ . Differentiating the finite sum gives  $y'_i = \sum_j (M'_{ij}v_j + M_{ij}v'_j)$ , hence  $y' = M'v + Mv'$ . Direct multiplication and differentiation give

$$y = \begin{pmatrix} t^2 + te^t \\ 2e^t \end{pmatrix}, \quad y' = \begin{pmatrix} 2t + (1+t)e^t \\ 2e^t \end{pmatrix}.$$

The product-rule route gives  $M'v = (e^t, 0)^T$  and  $Mv' = (2t + te^t, 2e^t)^T$ , confirming the same derivative.

**Step 2: Differentiate the inverse with the correct order.** The inverse is  $N = \begin{pmatrix} 1 & -t/2 \\ 0 & 1/2 \end{pmatrix}$ . Differentiating  $MN = I_2$  gives  $M'N + MN' = 0$ ; multiplying on the left by  $N$  yields

$$N' = -NM'N = \begin{pmatrix} 0 & -1/2 \\ 0 & 0 \end{pmatrix}.$$

This agrees with direct entrywise differentiation of  $N$ .

**Step 3: Test the square rule explicitly.** Since  $M^2 = \begin{pmatrix} 1 & 3t \\ 0 & 4 \end{pmatrix}$ ,

$$(M^2)' = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}}_{M'M} + \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{MM'}.$$

But  $2MM' = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ , which is different. The shortcut works when  $M$  and  $M'$  commute; they do not here.

**Step 4: Interpret the constant determinant.** The determinant is always 2, whereas  $M(t)(0, 1)^T = (t, 2)^T$  has length  $\sqrt{t^2 + 4}$ . Thus signed area scaling remains fixed while shear and some vector lengths change. Determinant information alone cannot determine every entry or every geometric effect of a matrix.