

Question 6

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

In three dimensions, let

$$u = (1, 1, 0)^T, \quad v = (1, 0, 1)^T, \quad w_\tau = (\tau, 1, 1)^T.$$

The plane of interest passes through $p = (1, 2, 3)^T$ and has direction vectors u, v . Use the scalar triple product $(u \times v) \cdot w$ as signed volume.

Tasks

1. Find a normal vector and a Cartesian equation for the plane. Verify that the specified point and directions satisfy the required conditions.
2. Find the area of the parallelogram spanned by u, v , and the signed and unsigned volumes spanned by u, v, w_τ .
3. Classify every τ for which the three direction vectors are linearly independent. At each dependent value, give an explicit nontrivial relation.
4. Explain how reversing the order of u, v affects the normal, plane equation, signed volume and unsigned volume. Distinguish a direction lying in the plane from a point lying on this translated plane.

Question 6 – Solution

Strategy. A cross product encodes both a perpendicular direction and an orientation. Translation affects a plane's points, not its direction space.

Step 1: Find and verify the normal. Direct calculation gives

$$n = u \times v = (1, -1, -1)^T.$$

Both dot products $n \cdot u$ and $n \cdot v$ vanish. The plane equation is $n \cdot (x - p) = 0$, or

$$x_1 - x_2 - x_3 = -4.$$

The point p satisfies it, and adding any combination of u, v to p leaves its left side unchanged.

Step 2: Compute area and volume. The parallelogram area is $\|n\| = \sqrt{3}$. The scalar triple product is

$$n \cdot w_\tau = \tau - 2, \quad \text{unsigned volume} = |\tau - 2|.$$

For example, $\tau = 0$ gives signed volume -2 and volume 2 .

Step 3: Prove the independence classification. If $\alpha u + \beta v + \gamma w_\tau = 0$, dotting with n gives $\gamma(\tau - 2) = 0$. For $\tau \neq 2$, this forces $\gamma = 0$. The second and third coordinates of $\alpha u + \beta v = 0$ then give $\alpha = \beta = 0$. Thus the vectors are independent exactly for $\tau \neq 2$. At $\tau = 2$, $w_2 = u + v$ is a nontrivial dependence, and the volume collapses to zero.

Step 4: Separate orientation from translation. Reversing the order gives $v \times u = -n$. The plane equation becomes $-x_1 + x_2 + x_3 = 4$, describing the same plane. Signed volume changes sign, while unsigned volume does not. As a direction, w_τ lies in the plane's direction space exactly when $n \cdot w_\tau = 0$, or $\tau = 2$. As a point with those coordinates, it lies on the translated plane exactly when $\tau - 2 = -4$, or $\tau = -2$. These are different questions because the plane does not pass through the origin.