

Question 5

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Let $a = (1, 2)^T$ and $b = (2, 1)^T$. We seek the point on the line through a closest to b . All points on that line have the form ta , $t \in \mathbb{R}$.

Tasks

1. Minimize $\|b - ta\|^2$ exactly, giving the unique closest point and the minimum distance.
2. Construct a matrix Q that sends any vector to its closest point on this same line. Compute Q^T and Q^2 .
3. Describe all vectors sent to zero and all vectors fixed by Q . Verify that the error $b - Qb$ is perpendicular to the line.
4. Prove $\|x\|^2 = \|Qx\|^2 + \|x - Qx\|^2$ for every x , and deduce a sharp length inequality with its equality condition.

Question 5 – Solution

Strategy. Complete the square to find the best coefficient, then express that coefficient as a linear function of the input.

Step 1: Minimize the distance. Using $a^T a = 5$, $a^T b = 4$ and $b^T b = 5$,

$$\|b - ta\|^2 = 5t^2 - 8t + 5 = 5(t - 4/5)^2 + 9/5.$$

The unique minimizer is $t = 4/5$, so

$$Qb = (4/5, 8/5)^T, \quad \text{minimum distance} = 3/\sqrt{5}.$$

Step 2: Build the projection matrix. For any input x , the minimizing coefficient is $(a^T x)/(a^T a)$. Therefore

$$Q = \frac{aa^T}{5} = \frac{1}{5} \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}, \quad Q^T = Q, \quad Q^2 = Q.$$

The last equality follows from $aa^T aa^T = a(a^T a)a^T = 5aa^T$.

Step 3: Describe the two perpendicular directions. The condition $Qx = 0$ is equivalent to $x_1 + 2x_2 = 0$, so the vectors sent to zero form the line through $(2, -1)^T$. The fixed vectors are exactly the multiples of a : the image lies on that line, and $Q(ta) = ta$. For the stated input, $b - Qb = (6/5, -3/5)^T$, whose dot product with a is zero.

Step 4: Prove the length identity and equality case. In general $a^T(x - Qx) = 0$, so Qx is perpendicular to $x - Qx$. Expanding the squared length of their sum gives

$$\|x\|^2 = \|Qx\|^2 + \|x - Qx\|^2, \quad \|Qx\| \leq \|x\|.$$

Equality holds exactly when $x - Qx = 0$, that is, when x lies on the line through a , including the zero vector. For b , the squared lengths are $5 = 16/5 + 9/5$. The plot uses equal axis scales and marks the right angle.

