

Question 4

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Use the ordered basis $u = (1, 1)^T$, $v = (1, -1)^T$ of the plane. Let P have columns u, v , so a coordinate column c represents the physical vector Pc . A transformation in standard coordinates is

$$T = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}.$$

Tasks

1. Find P^{-1} and the coordinates of $w = (3, 1)^T$ in the ordered basis.
2. Derive the matrix representing T in this basis, explaining the order of all three operations.
3. Apply the coordinate matrix to your coordinate column and check the result against direct standard-coordinate multiplication.
4. For arbitrary coordinate columns, derive the relation between coordinate length and physical length. Compare the determinants of the two representations and interpret the physical area factor.

Question 4 – Solution

Strategy. A vector and its coordinate column describe the same object in different languages. Convert in the correct order.

Step 1: Build and invert the basis matrix. Here

$$P = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad P^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad [w]_{u,v} = P^{-1}w = \boxed{(2, 1)^T}.$$

Thus $w = 2u + v$, as the figure illustrates.

Step 2: Express the transformation in the new coordinates. Start with a coordinate column c . Convert it to Pc , apply T , and convert the result back with P^{-1} . Consequently

$$C = P^{-1}TP = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 1 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}}.$$

Step 3: Verify both routes. The coordinate route gives $C(2, 1)^T = (4, 3)^T$, representing $P(4, 3)^T = (7, 1)^T$. Directly, $T(3, 1)^T = (7, 1)^T$ as well. Equivalently, $PC = TP$ verifies the agreement for every input, not only w .

Step 4: Compare lengths and area factors. Since $P^T P = 2I_2$,

$$\boxed{\|Pc\|^2 = c^T P^T P c = 2\|c\|^2.}$$

The basis vectors are perpendicular but have length $\sqrt{2}$, so physical length is $\sqrt{2}$ times coordinate length. Also $\det C = \det(P^{-1}) \det T \det P = \det T = 2$. The physical transformation doubles areas and preserves orientation, regardless of which basis is used to describe it. The negative determinant of P describes the basis orientation; it does not change this area factor.

