

Question 3

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Let

$$A_\tau = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \tau & 1 \\ 1 & 1 & \tau \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \tau \in \mathbb{R}.$$

Tasks

1. Find $\det A_\tau$ by determinant-preserving row operations and classify invertibility.
2. For the invertible cases, derive A_τ^{-1} by solving $A_\tau x = c$ for a general column c . Verify the formula.
3. Solve $A_\tau x = b$ for every τ . At the singular parameter, classify consistency for an arbitrary right-hand side c .
4. Compare the behavior near $\tau = 1$ for right-hand sides $(1, 2, 3)^T$ and $(1, 1, 1)^T$. Explain why invertibility for each nearby parameter does not imply a uniform bound on recovered vectors.

Question 3 – Solution

Strategy. Subtracting the first row exposes the parameter that controls both inversion and the exceptional case.

Step 1: Compute the determinant. Replace rows two and three by themselves minus row one. These operations do not change the determinant and produce

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & \tau - 1 & 0 \\ 0 & 0 & \tau - 1 \end{pmatrix}.$$

Therefore $\det A_\tau = (\tau - 1)^2$, and inversion is possible exactly when $\tau \neq 1$.

Step 2: Solve for a general right-hand side. Put $d = \tau - 1 \neq 0$. The subtracted equations give $x_2 = (c_2 - c_1)/d$, $x_3 = (c_3 - c_1)/d$, and $x_1 = c_1 - x_2 - x_3$. Thus

$$A_\tau^{-1} = \begin{pmatrix} 1 + 2/d & -1/d & -1/d \\ -1/d & 1/d & 0 \\ -1/d & 0 & 1/d \end{pmatrix}.$$

Substitution in the three original equations returns c_1, c_2, c_3 for every c , so multiplication $A_\tau A_\tau^{-1} = I_3$ is verified. The displayed matrices are symmetric, and transposing that equality also verifies multiplication in the opposite order.

Step 3: Handle the prescribed and singular systems. For $\tau \neq 1$,

$$x = \left(1 - \frac{3}{d}, \frac{1}{d}, \frac{2}{d} \right)^T.$$

At $\tau = 1$, all three left sides are $x_1 + x_2 + x_3$, while the entries of b differ, so there is no solution. For a general c , consistency at 1 is equivalent to $c_1 = c_2 = c_3$. If they coincide, the whole plane $x_1 + x_2 + x_3 = c_1$ is the solution set; otherwise it is empty.

Step 4: Distinguish invertibility from bounded recovery. For $b = (1, 2, 3)^T$, the coordinate $x_2 = 1/(\tau - 1)$ becomes unbounded near 1. For $c = (1, 1, 1)^T$, the unique nearby solution is always $(1, 0, 0)^T$. Thus some data remain harmless while other fixed data yield arbitrarily large solutions. The factors $1/d$ in the inverse prevent a uniform bound near the singular parameter, even though every matrix with $d \neq 0$ is invertible.