

Question 2

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

In the plane, apply the shear and quarter-turn

$$S = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Composition acts on column vectors from right to left.

Tasks

1. Compute RS and SR . Apply each to $v = (1, 1)^T$ and explain the difference in operation order.
2. Find every vector w for which $RSw = SRw$. Prove that your list is complete.
3. Find $(RS)^{-1}$ by reversing the operations, and verify the inverse by multiplication.
4. Map the four vertices of the unit square in their cyclic order under both products. Compare areas and orientation, and explain why equal determinants do not imply equal transformations.

Question 2 – Solution

Strategy. Matrix multiplication records the order of geometric operations. Determinants retain only part of that information.

Step 1: Compute both compositions. Direct multiplication gives

$$RS = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}, \quad SR = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}.$$

Thus $RSv = (-1, 3)^T$ and $SRv = (1, 1)^T$. The first shears then rotates; the second rotates then shears.

Step 2: Test agreement on an arbitrary vector. Subtracting gives

$$(RS - SR)w = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

This is zero exactly when $w_1 = w_2 = 0$. Therefore the two compositions agree only on the zero vector, despite each being invertible.

Step 3: Undo the operations in reverse order. Since $S^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ and $R^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

$$(RS)^{-1} = S^{-1}R^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Multiplication with RS on either side gives I_2 .

Step 4: Compare the images of the square. For vertices $(0, 0), (1, 0), (1, 1), (0, 1)$ in that order, the images are

$$\begin{array}{l|llll} RS & (0, 0) & (0, 1) & (-1, 3) & (-1, 2) \\ SR & (0, 0) & (2, 1) & (1, 1) & (-1, 0) \end{array}$$

Both determinants are 1, so both images have area 1 and preserve the counterclockwise orientation. The parallelograms nevertheless differ. A determinant encodes signed area scaling, not the full image of each vector. The plot uses equal scales on both axes.

