

Question 1

Work with real matrices and column vectors. Write I_n for the $n \times n$ identity, A^T for transpose, and $\|v\| = \sqrt{v^T v}$ for Euclidean length. Show the reasoning behind every classification; do not use eigenvalue methods.

Consider the rectangular matrices

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Tasks

1. Check the dimensions of AB and BA , compute both products, and identify which identity matrix occurs.
2. For an arbitrary $b = (b_1, b_2)^T$, describe every solution of $Ax = b$ and explain which solution Bb selects.
3. Prove that no matrix C can satisfy $CA = I_3$. Use a nonzero vector sent to zero by A , rather than only counting equations.
4. For $P = BA$, prove $P^2 = P$, describe all vectors fixed by P , and compute Pv for $v = (2, -1, 3)^T$. Explain what information this operation discards.

Question 1 – Solution

Strategy. A rectangular matrix can have a right inverse without having a left inverse. Check what happens to vectors, not just products.

Step 1: Multiply with dimensions attached. The sizes are $A : 2 \times 3$ and $B : 3 \times 2$. Thus

$$AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2, \quad BA = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \neq I_3.$$

The two products even act on spaces of different dimensions.

Step 2: Recover the whole solution family. The equation $Ax = b$ says $x_1 + x_3 = b_1$ and $x_2 + x_3 = b_2$. Hence

$$x = (b_1 - t, b_2 - t, t)^T, \quad t \in \mathbb{R}.$$

Every such vector works, and every solution has this form. The vector $Bb = (b_1, b_2, 0)^T$ selects $t = 0$; it is a choice, not the only inverse image.

Step 3: Rule out a left inverse. The nonzero vector $n = (-1, -1, 1)^T$ satisfies $An = 0$. If $CA = I_3$, then $n = I_3n = CAn = C0 = 0$, a contradiction. Thus no left inverse exists, even though $AB = I_2$.

Step 4: Interpret the repeated operation. Associativity gives $P^2 = B(AB)A = BI_2A = P$. For $x = (x_1, x_2, x_3)^T$, $Px = (x_1 + x_3, x_2 + x_3, 0)^T$. Consequently $Px = x$ exactly when $x_3 = 0$, and

$$P(2, -1, 3)^T = (5, 2, 0)^T.$$

The discarded difference is $x - Px = x_3(-1, -1, 1)^T$, invisible to A . Indeed $APx = Ax$. This is an idempotent projection onto the plane $x_3 = 0$, but it is not the perpendicular projection: it changes the first two coordinates.