

Question 8

An ideal temperature controller is modeled in dimensionless variables. The room temperature is 0, the target is 1, and the scaled heater output is $P(t)$. The plant obeys $T' = -T + P$. A controller with memory sets

$$P(t) = k \int_0^t [1 - T(\tau)] d\tau, \quad T(0) = 0, \quad k > 0.$$

There is no heater saturation, delay or heat-source noise in this ideal model.

Tasks

1. Convert the plant and controller into a two-state autonomous IVP. Find its equilibrium and explain why P is a state rather than a fixed parameter.
2. Shift to the equilibrium and show it attracts every initial state for every $k > 0$. Classify the eigenvalues at the critical gain.
3. For $k = 1/4$, solve the specified IVP. Prove both T and P stay in $[0, 1)$ at finite times and increase toward 1. Sketch them.
4. Replace the memory controller by the proportional rule $P = k(1 - T)$, with the same $k > 0$. Find its limiting temperature and explain the difference in steady error and the practical scope of the ideal memory model.

Question 8 – Solution

Strategy. The accumulated error supplies an additional state; its equilibrium condition eliminates the temperature offset of proportional feedback.

Step 1: Introduce the controller state. Differentiation gives

$$T' = -T + P, \quad P' = k(1 - T), \quad (T(0), P(0)) = (0, 0).$$

At equilibrium, $T = 1$ and $P = 1$. Heater output depends on accumulated past error, so two states with the same temperature can have different future motion if their current P values differ.

Step 2: Check stability and the critical gain. For $x = T - 1$, $u = P - 1$, the matrix is $\begin{pmatrix} -1 & 1 \\ -k & 0 \end{pmatrix}$. Its roots are $(-1 \pm \sqrt{1 - 4k})/2$. For $0 < k < 1/4$ both are real and negative; at $k = 1/4$ there is a repeated root $-1/2$; for $k > 1/4$ they are complex with real part $-1/2$. Every homogeneous mode decays, including the polynomial times exponential at the repeated root. Thus the equilibrium attracts every initial state within this ideal linear model.

Step 3: Solve the critical-gain response. Eliminating P gives $T'' + T' + T/4 = 1/4$, with $T(0) = T'(0) = 0$. The repeated-root solution and $P = T' + T$ yield

$$T = 1 - (1 + t/2)e^{-t/2}, \quad P = 1 - (1 + t/4)e^{-t/2}.$$

The initial values vanish and $T' = te^{-t/2}/4 \geq 0$, $P' = (1/4 + t/8)e^{-t/2} > 0$. The latter equals $(1 - T)/4$, checking the controller equation. The positive subtracted terms and monotonicity imply $0 \leq T, P < 1$ at finite t , and both tend to 1. Here the heater output stays nonnegative without needing a saturation rule.

Step 4: Compare steady errors. Proportional feedback instead gives $T' = k - (1 + k)T$, so $T \rightarrow k/(1 + k)$ and the steady error is $1/(1 + k) > 0$. The memory controller can retain $P = 1$ even when the current error is zero; a proportional controller would then set $P = 0$. Real actuator limits and delays can change stability or feasibility, so the ideal conclusions apply under the stated assumptions, not to every physical controller.

