

Question 7

A continuous two-stage population model uses juvenile and adult abundances J, A , measured in thousands, with time in years. Juveniles mature at per-capita rate 1 yr^{-1} and die at rate 1 yr^{-1} . Adults die at rate 1 yr^{-1} and produce juveniles at per-capita rate $b \text{ yr}^{-1}$, where $b \geq 0$. Assume no crowding and strictly positive initial J_0, A_0 .

Tasks

1. Derive the stage equations and explain why maturation must appear with opposite signs in the two balances.
2. Find the eigenvalues and classify long-time extinction, a critical threshold, and unbounded growth as b varies. Justify that the growing mode is actually present for the stated initial data.
3. At the critical value, find a conserved weighted total, the exact solution and its limiting stage abundances.
4. For every $b \geq 0$, find the limiting ratio J/A and interpret it. Explain why this linear model cannot determine a realistic carrying capacity.

Question 7 – Solution

Strategy. Separate transfers between stages from births and deaths, then use the dominant mode and the exceptional zero eigenvalue.

Step 1: Construct the stage balances. Maturation removes juveniles and adds adults, whereas juvenile death removes individuals without adding to the adult stage. Therefore $J' = bA - 2J$, $A' = J - A$. All coefficients have units yr^{-1} . Nonnegative off-diagonal rates preserve the nonnegative quadrant; the strictly positive initial populations remain positive.

Step 2: Determine the growth threshold. The characteristic equation is $\lambda^2 + 3\lambda + 2 - b = 0$, so $\lambda_{\pm} = (-3 \pm \sqrt{1 + 4b})/2$. Put $d = \sqrt{1 + 4b}$. The adult solution has dominant coefficient

$$A(t) = c_+ e^{\lambda_+ t} + c_- e^{\lambda_- t}, \quad c_+ = \frac{J_0 + (1 + d)A_0/2}{d} > 0.$$

Thus both populations tend to zero for $0 \leq b < 2$, approach a nonzero critical equilibrium for $b = 2$, and grow without bound for $b > 2$. For $b > 2$, the dominant juvenile coefficient $(\lambda_+ + 1)c_+$ is also positive.

Step 3: Solve the critical case exactly. When $b = 2$, $K = J + 2A = J_0 + 2A_0$ is conserved. Since $A' = K - 3A$,

$$A = K/3 + (A_0 - K/3)e^{-3t}, \quad J = K/3 - 2(A_0 - K/3)e^{-3t}.$$

Both tend to $K/3$ thousand. The critical equilibria form the line $J = A$; there is no single equilibrium amount independent of initial data.

Step 4: Interpret the asymptotic stage ratio. Since $J = A' + A$ and $c_+ > 0$, division by the dominant adult mode gives

$$\lim_{t \rightarrow \infty} \frac{J}{A} = \lambda_+ + 1 = \frac{\sqrt{1 + 4b} - 1}{2}.$$

For $b = 0$, this limit is zero: juveniles decay at rate 2, adults at rate 1. The ratio describes relative stage composition even when the entire population tends to zero; it does not imply constant abundance. For $b > 2$ there is no crowding term to halt growth. A carrying capacity requires additional density-dependent assumptions absent from this model.