

Question 5

Two magnetically coupled loops have inductances $L_1 = L_2 = 2$ H, mutual inductance $M = 1$ H, and resistances $R_1 = R_2 = 1$ Ω . Reference current directions and coil orientations are chosen so the mutual derivative terms have positive signs in both voltage-drop balances. At $t = 0$ a constant 3 V source is connected to loop 1; loop 2 has no source. Initially both currents are zero. Time is in seconds; the ideal linear circuit model applies.

Tasks

1. Use Kirchhoff's voltage law to derive the coupled derivative equations, then solve them for i'_1, i'_2 . Explain why the inductance matrix is invertible.
2. Solve the initial-value problem using current sum and difference, and find the limiting currents.
3. Determine the sign of $i_2(t)$ for $t > 0$ and its most negative value and time. Explain what a negative current means here, and sketch both currents.
4. With magnetic energy $W = (2i_1^2 + 2i_1i_2 + 2i_2^2)/2$, derive the power balance. Explain what changes when both voltage sources are zero.

Question 5 – Solution

Strategy. Invert the inductance matrix, not just its diagonal; the sum and difference currents see different inductances.

Step 1: Form the coupled voltage balances. Voltage drops give $2i_1' + i_2' = 3 - i_1$ and $i_1' + 2i_2' = -i_2$. The inductance determinant is $4 - 1 = 3 \text{ H}^2 > 0$, so

$$i_1' = 2 - \frac{2}{3}i_1 + \frac{1}{3}i_2, \quad i_2' = -1 + \frac{1}{3}i_1 - \frac{2}{3}i_2.$$

Each right side has units A/s. Ignoring the mutual terms would predict $i_2 \equiv 0$ and would fail the second voltage balance during the transient.

Step 2: Solve the sum and difference. Let $s = i_1 + i_2$, $d = i_1 - i_2$. Adding and subtracting the voltage equations gives $3s' = 3 - s$ and $d' = 3 - d$, with $s(0) = d(0) = 0$. Thus $s = 3(1 - e^{-t/3})$, $d = 3(1 - e^{-t})$, and

$$i_1 = 3 - \frac{3}{2}(e^{-t/3} + e^{-t}), \quad i_2 = \frac{3}{2}(e^{-t} - e^{-t/3}).$$

They tend to (3, 0) A, the resistive steady state. Their initial slopes are (2, -1) A/s, agreeing with the inverted model.

Step 3: Locate the induced-current extremum. For $t > 0$, $e^{-t} < e^{-t/3}$, so $i_2 < 0$. Its derivative vanishes when $e^{-t/3}/3 = e^{-t}$, at $t_* = (3/2) \log 3$ s. The derivative changes from negative to positive, giving

$i_{2,\min} = -1/\sqrt{3}$ A. The sign means actual current flows opposite its chosen reference direction; it is not a negative stored energy or a violation of the circuit model.

Step 4: Verify the power identity. Writing $L = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $W' = i^T L i' = i^T (V - i)$ gives $W' = 3i_1 - i_1^2 - i_2^2$ in watts. With $V = 0$ instead, $W' = -i_1^2 - i_2^2 \leq 0$. Since L has positive eigenvalues 1, 3, $W > 0$ for nonzero currents. The driven circuit can gain magnetic energy from its source; the source-free circuit can only dissipate it.

