

Question 1

Two perfectly mixed tanks contain 100 L and 200 L of solution. Two pumps transfer 10 L/min in each direction; there is no external inflow, outflow, or reaction. Initially the first tank contains 10 kg of salt and the second contains none. Let $x(t), y(t)$ be their salt amounts in kg, with t in minutes.

Tasks

1. Derive the amount equations from salt fluxes, explaining why the volumes stay constant and why the two transfer coefficients differ.
2. Find a conserved quantity and the equilibrium amounts. Does equilibrium mean equal amounts or equal concentrations?
3. Solve the initial-value problem and verify nonnegativity and conservation for all $t \geq 0$.
4. Find the earliest time at which the concentration difference is at most 0.005 kg/L, and show it stays below this tolerance thereafter. Sketch the two concentrations and their common limiting level.

Question 1 – Solution

Strategy. Use volume flow times concentration for each salt flux, then use conservation to reduce the coupled balances.

Step 1: Construct the two balances. Each tank gains and loses 10 L/min, so its volume is fixed. Its outgoing salt rate is 10 times its own concentration. Thus

$$x' = -\frac{1}{10}x + \frac{1}{20}y, \quad y' = \frac{1}{10}x - \frac{1}{20}y.$$

The coefficients have units min^{-1} : equal volume flows remove different fractions of the unequal tank volumes per minute.

Step 2: Use the conserved total. Adding gives $(x + y)' = 0$, so $x + y = 10$ kg. At equilibrium $x/100 = y/200$. Consequently $(x_*, y_*) = (10/3, 20/3)$ kg and the common concentration is $1/30$ kg/L. Equal amounts would leave unequal concentrations and hence a nonzero net salt transfer.

Step 3: Solve and check the physical range. Substituting $y = 10 - x$ gives $x' = 1/2 - 3x/20$. The initial data yield

$$x = \frac{10}{3} + \frac{20}{3}e^{-3t/20}, \quad y = \frac{20}{3}(1 - e^{-3t/20}).$$

Their sum is 10, both are nonnegative, and their initial values are $(10, 0)$. Differentiation gives $x' = -e^{-3t/20}$ and $y' = e^{-3t/20}$ kg/min; substitution into the original flux balances gives the same rates.

Step 4: Compute a lasting tolerance time. The concentration difference is

$$\frac{x}{100} - \frac{y}{200} = \frac{1}{10}e^{-3t/20} > 0.$$

It decreases strictly, so its first value 0.005 occurs at $t = \frac{20}{3} \log 20 \text{ min} \approx 19.97 \text{ min}$. The inequality holds for this time and every later time. The first concentration approaches the dotted limit from above; the second approaches it from below.

