

Question 9

A rectangular input drives an initially resting oscillator:

$$x' = y, \quad y' = -4x + u_T(t), \quad x(0) = y(0) = 0, \quad u_T(t) = H(t) - H(t - T), \quad T > 0.$$

Use the causal step convention for $t \geq 0$. States are continuous at T , and equations at the switch are understood one-sidedly.

Tasks

1. Find both Laplace transforms and express the response as an undelayed step response minus its delayed copy.
2. Calculate $(x(T), y(T))$ and determine exactly which positive pulse durations leave the state at rest forever after the pulse ends.
3. For arbitrary T , calculate the constant value of $y^2 + 4x^2$ after switch-off. Relate complete cancellation to zeros of the input transform at the oscillator poles.
4. Compare $T = \pi$ and $T = \pi/2$: give their post-pulse responses and sketch $x(t)$ on $0 \leq t \leq 2\pi$. Explain why a nonzero, nonnegative pulse can leave no residual oscillation.

Question 9 – Solution

Strategy. Shift a single step response, then check both state components at switch-off; displacement alone is not a stopping condition.

Step 1: Transform and invert by shifting. Since $sF = G$ and $sG = -4F + (1 - e^{-Ts})/s$,

$$F = \frac{1 - e^{-Ts}}{s(s^2 + 4)}, \quad G = \frac{1 - e^{-Ts}}{s^2 + 4}, \quad \text{Re } s > 0.$$

Let $h(t) = (1 - \cos 2t)/4$ and $j(t) = \sin 2t/2$. Then

$$x = h(t) - H(t - T)h(t - T), \quad y = j(t) - H(t - T)j(t - T).$$

Both delayed terms begin at zero, proving state continuity.

Step 2: Enforce complete rest at the endpoint. At T , the state is $((1 - \cos 2T)/4, \sin 2T/2)^T$. Both components vanish exactly when $\cos 2T = 1$, giving $T = k\pi$, $k = 1, 2, \dots$; the earliest is π . Zero endpoint data then yield the identically zero unforced continuation. Conversely, rest for all later times requires these endpoint values by continuity.

Step 3: Compute the residual and its spectral cancellation. After the pulse, differentiation gives $(y^2 + 4x^2)' = 2y(-4x) + 8xy = 0$. Evaluation at T yields

$$y^2 + 4x^2 = \frac{1}{4} \sin^2 2T + \frac{1}{4} (1 - \cos 2T)^2 = \sin^2 T.$$

The finite-pulse transform $U_T = (1 - e^{-Ts})/s$ has zeros at $s = \pm 2i$ exactly when $e^{2iT} = 1$. These are precisely the stopping durations; they cancel the oscillator poles. At $s = 0$ its singularity is removable, with value T , so a nonzero total input area does not prevent this cancellation.

Step 4: Compare two durations. For $T = \pi$, $x = y = 0$ after π . For $T = \pi/2$, writing $\tau = t - \pi/2 \geq 0$, the endpoint state is $(1/2, 0)$ and $x = \frac{1}{2} \cos 2\tau$, $y = -\sin 2\tau$ afterward. During either pulse, $x = h(t)$ and $y = j(t)$; the plotted curves coincide up to $\pi/2$. A full-period pulse lets the forced trajectory return to rest before switch-off; accumulated state contributions have different oscillatory phases and can cancel even when the input never becomes negative. Equal input sign is not equal state phase.

