

Question 8

For $\varepsilon \geq 0$, let

$$x' = -\varepsilon x - y + 1, \quad y' = x - \varepsilon y, \quad x(0) = y(0) = 0, \quad t \geq 0.$$

A student evaluates $\lim_{s \rightarrow 0^+} s\mathcal{L}\{x\}$ and $\lim_{s \rightarrow 0^+} s\mathcal{L}\{y\}$ and declares these the limiting state for every ε . For rational transforms, the final-value theorem requires all poles of the simplified $s\mathcal{L}\{f\}$ to have negative real part.

Tasks

1. Find both transformed components for general ε and calculate the student's two proposed limits.
2. Solve the case $\varepsilon = 0$. Check the theorem's hypothesis and determine whether either component has a limit as $t \rightarrow \infty$.
3. For $\varepsilon > 0$, invert the transforms and establish the true limiting state. Explain why the theorem now applies.
4. Show that $y_\varepsilon \rightarrow y_0$ at each fixed time as $\varepsilon \rightarrow 0^+$ but not uniformly on $[0, \infty)$. Use times $2\pi n$ to obtain a lower bound on the uniform error. Sketch y_0 and $y_{1/2}$.

Question 8 – Solution

Strategy. Check pole locations before applying the final-value theorem, and distinguish fixed-time convergence from an infinite-time assertion.

Step 1: Solve the transformed system. With $D = (s + \varepsilon)^2 + 1$, the equations give

$$F = \frac{s + \varepsilon}{sD}, \quad G = \frac{1}{sD}.$$

A sufficient convergence half-plane is $\operatorname{Re} s > 0$. The proposed final values are $\varepsilon/(1 + \varepsilon^2)$ and $1/(1 + \varepsilon^2)$.

Step 2: Examine the undamped case. At $\varepsilon = 0$, inversion yields $x = \sin t$, $y = 1 - \cos t$. Neither has a limit: the values along successive maxima and minima differ. Both sF and sG have poles at $\pm i$, so the stated theorem's hypothesis fails. The finite algebraic values 0 and 1 do not prove convergence in time.

Step 3: Resolve the damped response. For $\varepsilon > 0$, decompose, for example,

$$G = \frac{1}{1 + \varepsilon^2} \left(\frac{1}{s} - \frac{s + \varepsilon}{D} - \frac{\varepsilon}{D} \right).$$

Inverting this and the analogous decomposition of F gives

$$x_\varepsilon = \frac{\varepsilon - e^{-\varepsilon t}(\varepsilon \cos t - \sin t)}{1 + \varepsilon^2}, \quad y_\varepsilon = \frac{1 - e^{-\varepsilon t}(\cos t + \varepsilon \sin t)}{1 + \varepsilon^2}.$$

Both start at zero. The exponentially decaying terms leave the limit $(\varepsilon, 1)^T/(1 + \varepsilon^2)$. Now the poles of sF and sG are $-\varepsilon \pm i$, strictly in the left half-plane, as required. The limit also solves the equilibrium equations.

Step 4: Test uniformity on the whole half-line. At fixed t , the formula gives $y_\varepsilon(t) \rightarrow 1 - \cos t = y_0(t)$. But $y_0(2\pi n) = 0$, whereas $y_\varepsilon(2\pi n) = (1 - e^{-2\pi n \varepsilon})/(1 + \varepsilon^2)$. Consequently

$$\sup_{t \geq 0} |y_\varepsilon(t) - y_0(t)| \geq \frac{1}{1 + \varepsilon^2}.$$

This lower bound tends to 1, not 0. Fixed-time convergence therefore cannot be used to pass a limiting-state conclusion to the undamped system. The dotted guide is the damped limit $4/5$ for $\varepsilon = 1/2$.

