

Question 5

Consider, for $t \geq 0$,

$$Z' = AZ + B, \quad A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad Z(0) = \begin{pmatrix} p \\ q \end{pmatrix}.$$

A student cancels a factor $s - 2$ in the zero-initial-state transform and concludes that every solution of this forced system is bounded.

Tasks

1. Compute $(sI - A)^{-1}B$ and find the response when $p = q = 0$ using Laplace transforms.
2. Explain the cancellation using the direction of B , and identify the system mode that the constant input does not excite from zero.
3. Retain the initial-state term in the transformed system and determine exactly which (p, q) produce bounded solutions on $[0, \infty)$.
4. Perturb the zero initial state to $(\varepsilon, 2\varepsilon)^T$, with $\varepsilon \neq 0$. Find the exact difference from the zero-state response and assess the student's conclusion.

Question 5 – Solution

Strategy. Separate the input contribution from the initial-state contribution before cancelling rational factors.

Step 1: Calculate the zero-state response. The determinant is $s^2 - s - 2 = (s - 2)(s + 1)$, and

$$(sI - A)^{-1}B = \frac{1}{(s - 2)(s + 1)} \begin{pmatrix} s - 1 & 1 \\ 2 & s \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{s + 1} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The constant input contributes an additional $1/s$. Inverting yields $\boxed{Z_0(t) = (1 - e^{-t})(1, -1)^T}$, which is bounded. Its transform converges for $\operatorname{Re} s > 0$; the cancelled expression extends through $s = 2$, where the full matrix resolvent itself is undefined.

Step 2: Identify why the pole disappears. We have $AB = -B$, so the input lies entirely in the stable eigendirection. The other eigenpair is $2, (1, 2)^T$. From zero, this input never excites that direction. The pole cancellation describes the chosen input and data; it does not change either eigenvalue of A .

Step 3: Include all initial conditions. Write $(p, q)^T = c_+(1, 2)^T + c_-(1, -1)^T$, where $c_+ = (p + q)/3$ and $c_- = (2p - q)/3$. The transform contains

$$\frac{c_+}{s - 2} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{c_-}{s + 1} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{s(s + 1)} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore

$$Z = c_+ e^{2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + [1 + (c_- - 1)e^{-t}] \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

The growing vector cannot be cancelled by a bounded term, so boundedness holds exactly when $\boxed{p + q = 0}$. Then the limit is $(1, -1)^T$. A common convergence half-plane for the general family is $\operatorname{Re} s > 2$.

Step 4: Test sensitivity to the omitted term. For the specified perturbation, $c_+ = \varepsilon$ and $c_- = 0$. The exact difference is $\boxed{Z - Z_0 = \varepsilon e^{2t}(1, 2)^T}$, whose Euclidean norm is $\sqrt{5}|\varepsilon|e^{2t} \rightarrow \infty$. Thus arbitrarily small initial perturbations in that direction destroy boundedness. The student's inference confuses a bounded zero-state response with boundedness for all initial states.