

Question 3

A delayed input drives the zero-state system

$$x' = -x + H(t - 2), \quad y' = x - 2y, \quad x(0) = y(0) = 0, \quad t \geq 0.$$

Here $H(t - a)$ is zero for $t < a$ and one for $t > a$; its value at the switch does not affect the continuous state. Interpret derivatives there one-sidedly.

Tasks

1. Find the transforms of both components and identify the factor that encodes the input delay.
2. Invert them to obtain a piecewise solution. Explain why multiplying an undelayed response by e^{-2t} does not implement a delay.
3. Determine continuity of x, y, x', y' at $t = 2$, and find the one-sided values of y'' there.
4. Prove both components increase after the switch and approach their limits without overshoot. Sketch their time responses, marking the switch and limiting levels.

Question 3 – Solution

Strategy. First invert the undelayed rational factors, then translate the entire response using the second shifting theorem.

Step 1: Transform both equations. With $F = \mathcal{L}\{x\}$ and $G = \mathcal{L}\{y\}$, $(s+1)F = e^{-2s}/s$ and $(s+2)G = F$. Hence

$$F = \frac{e^{-2s}}{s(s+1)}, \quad G = \frac{e^{-2s}}{s(s+1)(s+2)}, \quad \operatorname{Re} s > 0.$$

It is e^{-2s} , rather than an exponential in time, that encodes the delay.

Step 2: Shift the inverse transforms. The undelayed inverses are $1 - e^{-t}$ and $\frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t} = \frac{1}{2}(1 - e^{-t})^2$. Writing $\tau = t - 2$, the solution is zero for $t < 2$, and for $t \geq 2$,

$$x = 1 - e^{-\tau}, \quad y = \frac{1}{2}(1 - e^{-\tau})^2.$$

These formulas give zero at the joining point. Multiplication by e^{-2t} would change decay rates while generally leaving a nonzero response before $t = 2$; it is a different transform operation.

Step 3: Check the switch regularity. Both states are continuous. Before the switch their derivatives vanish; after it $x' = e^{-\tau}$ and $y' = e^{-\tau} - e^{-2\tau}$. Thus $x'(2^-) = 0$, $x'(2^+) = 1$, whereas $y'(2^-) = y'(2^+) = 0$. Afterward $y'' = -e^{-\tau} + 2e^{-2\tau}$, so $y''(2^-) = 0$, $y''(2^+) = 1$. In particular y is continuously differentiable at the switch but is not twice differentiable there.

Step 4: Establish shape and limits. For $\tau > 0$, $x' > 0$ and $y' = e^{-\tau}(1 - e^{-\tau}) > 0$. Also $0 < x < 1$ and $0 < y < 1/2$, with limits 1 and $1/2$. These inequalities prove the absence of overshoot for all later times. The dotted horizontal guides in the sketch are limits; the solid and dashed curves are the components. Both remain zero throughout the waiting interval.

