

Question 2

Let $Z = (x, y, z)^T$ satisfy $Z' = AZ$ for $t \geq 0$, where

$$A = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{pmatrix}, \quad Z(0) = (a, b, c)^T.$$

The characteristic polynomial has a triple root. Investigate how the initial vector determines the poles actually present in each transformed component.

Tasks

1. Compute $(sI - A)^{-1}$ directly, and verify its product with $sI - A$ is I . State where the inverse matrix exists.
2. Use the resolvent to find all three Laplace transforms and invert them for arbitrary real a, b, c .
3. For the initial state $(0, 0, 1)^T$, verify the original equations and explain the different polynomial degrees in the three components.
4. Classify the pole order of $\mathcal{L}\{x\}$ at $s = -2$ for every (a, b, c) , including the case of no pole. Does a triple characteristic root force a triple pole in every component?

Question 2 – Solution

Strategy. Invert the triangular matrix before substituting data; its powers of $(s+2)^{-1}$ display how a chain transmits the initial state.

Step 1: Compute the resolvent. Put $q = s + 2$. For $q \neq 0$,

$$(sI - A)^{-1} = \begin{pmatrix} q^{-1} & q^{-2} & q^{-3} \\ 0 & q^{-1} & q^{-2} \\ 0 & 0 & q^{-1} \end{pmatrix}.$$

Multiplication by the upper triangular matrix with diagonal q and superdiagonal -1 gives diagonal entries 1; each off-diagonal entry cancels. The inverse exists exactly for $s \neq -2$. This algebraic domain is larger than the convergence half-plane $\operatorname{Re} s > -2$ for the nonzero responses.

Step 2: Multiply by the initial vector. The transforms are $a/q + b/q^2 + c/q^3$, $b/q + c/q^2$, and c/q . Since $\mathcal{L}\{t^n e^{-2t}\} = n!/(s+2)^{n+1}$,

$$x = e^{-2t}(a + bt + ct^2/2), \quad y = e^{-2t}(b + ct), \quad z = ce^{-2t}.$$

The factorial is essential: the inverse of q^{-3} is $t^2 e^{-2t}/2$.

Step 3: Check transmission along the chain. For the specified initial state,

$$Z = (t^2 e^{-2t}/2, te^{-2t}, e^{-2t})^T.$$

Directly, $z' = -2z$, $y' = -2y + z$ and $x' = -2x + y$, with the required initial state. After multiplication by e^{2t} , the equations become $(e^{2t}z)' = 0$, $(e^{2t}y)' = e^{2t}z$ and $(e^{2t}x)' = e^{2t}y$. Two successive integrations produce the quadratic factor.

Step 4: Classify actual pole orders. The numerator of the first transform over q^3 is $aq^2 + bq + c$. Its order at $q = 0$ gives the exhaustive classification:

initial-data condition	pole order of $\mathcal{L}\{x\}$
$c \neq 0$	3
$c = 0, b \neq 0$	2
$c = b = 0, a \neq 0$	1
$a = b = c = 0$	no pole

Thus the characteristic multiplicity supplies a possible order, not an order every response must attain. Even for $c \neq 0$, the third component has only a simple pole, whereas the first has a triple pole.