

Question 1

For $t \geq 0$, consider the coupled initial-value problem

$$x' = -2x + y, \quad y' = 2x - 3y, \quad x(0) = 1, \quad y(0) = 0.$$

Write $F(s) = \mathcal{L}\{x\}(s)$ and $G(s) = \mathcal{L}\{y\}(s)$, using one-sided Laplace transforms. Recover both components by solving the transformed algebraic system.

Tasks

1. Derive the two transformed equations, retaining the initial terms. Solve for F and G and state a common half-plane of convergence.
2. Invert both transforms and verify the initial values and original differential equations.
3. Find the transform and time response of $w = x - y$. Explain why one pole of the full system is absent from this observation.
4. Find the exact time and value of the largest $y(t)$ on $t \geq 0$. Explain how y can initially increase while both system modes decay.

Question 1 – Solution

Strategy. Keep the initial data on the right side of the transformed system, then distinguish component responses from observed combinations.

Step 1: Solve the algebraic system. The derivative rule gives $(s+2)F - G = 1$ and $-2F + (s+3)G = 0$. Their determinant is $(s+1)(s+4)$, so

$$F = \frac{s+3}{(s+1)(s+4)}, \quad G = \frac{2}{(s+1)(s+4)}.$$

Both transforms converge for $\operatorname{Re} s > -1$. Dropping the initial term 1 would incorrectly give the zero solution.

Step 2: Invert and check. Partial fractions give

$$x = \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t}, \quad y = \frac{2}{3}(e^{-t} - e^{-4t}).$$

At zero these are $(1, 0)$. Differentiation gives $x' = -\frac{2}{3}e^{-t} - \frac{4}{3}e^{-4t}$ and $y' = -\frac{2}{3}e^{-t} + \frac{8}{3}e^{-4t}$, which equal $-2x + y$ and $2x - 3y$.

Step 3: Explain the cancelled pole. Subtracting the transforms gives $F - G = 1/(s+4)$, hence $w = e^{-4t}$. The slow mode is a multiple of $(1, 1)^T$; subtracting its components annihilates it. The state still contains this mode. Cancellation in an observation does not remove a mode from the system.

Step 4: Locate the component peak. Since $y' = \frac{2}{3}e^{-t}(4e^{-3t} - 1)$, it changes sign once, from positive to negative, at $t_* = \log(4)/3$. Thus

$$\max_{t \geq 0} y(t) = \frac{1}{2} 4^{-1/3}, \quad t_* = \log(4)/3.$$

Here $y'(0) = 2$ because the first component feeds the second. Decay of each mode does not imply monotonic decay of every component: their signed combination initially grows. The curve tends back to zero.

