

Question 10

Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$. Consider

$$X' = JX + e^{-t^2} R(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad t \geq 0.$$

You may use $\int_0^\infty e^{-s^2} ds = \sqrt{\pi}/2$. Exact definite-integral answers are expected; no elementary antiderivative is assumed.

Tasks

1. Use variation of parameters to find the solution for arbitrary $X(0) = c$. Specialize to $c = 0$.
2. For $c = 0$, find the limiting norm. Does the state converge to a point or become periodic? Explain why forcing that tends to zero need not produce a state that tends to zero.
3. Find exactly the initial state that makes $X(t) \rightarrow 0$. Express that solution with an integral from t to infinity.
4. Prove a quantitative decay bound for this selected solution using $\int_t^\infty e^{-s^2} ds \leq e^{-t^2}/(2t)$ for $t > 0$, deriving the inequality as part of the solution.

Question 10 – Solution

Strategy. A rotating frame converts the forcing into accumulated scalar amplitude; cancelling its total integral removes the persistent free rotation.

Step 1: Integrate in rotating coordinates. Since $R' = JR$ and $R^{-1} = R(-t)$, writing $X = R(t)Z$ gives $Z' = e^{-t^2}(1, 0)^T$. Define $G(t) = \int_0^t e^{-s^2} ds$. Then

$$X(t) = R(t) \begin{bmatrix} c + G(t) \\ 0 \end{bmatrix}.$$

For $c = 0$ the state is $G(t)(\cos t, \sin t)^T$. Differentiation verifies the equation, including the non-elementary amplitude derivative.

Step 2: Interpret the zero-state asymptotics. Let $L = \sqrt{\pi}/2$. For $c = 0$, the norm is $G(t) \uparrow L$, strictly increasing at every finite time, so the solution is not periodic. At times $2\pi n$ and $2\pi n + \pi$ it tends to $(L, 0)^T$ and $(-L, 0)^T$, so no point limit exists. The input decays, but the undamped homogeneous rotation retains the accumulated amplitude; there is no dissipative mechanism to erase it.

Step 3: Select the unique cancelling initial state. Because rotations preserve length, $\|X(t)\| \rightarrow \|c + L(1, 0)^T\|$. Thus $X(t) \rightarrow 0$ exactly when $c = (-L, 0)^T$. For that state,

$$X(t) = -R(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \int_t^\infty e^{-s^2} ds.$$

Any other initial state leaves a nonzero limiting amplitude.

Step 4: Bound the remaining tail. For $s \geq t > 0$, $1 \leq s/t$. Therefore

$$0 \leq \int_t^\infty e^{-s^2} ds \leq \frac{1}{t} \int_t^\infty se^{-s^2} ds = \frac{e^{-t^2}}{2t}.$$

The selected solution consequently obeys $\|X(t)\| \leq e^{-t^2}/(2t) \quad (t > 0)$. This is a tail estimate, not a formula at $t = 0$; the initial norm there is L . The definite integral and the bound avoid inventing an elementary antiderivative or replacing the exact response by numerical samples.