

Question 8

For $t \geq 0$, let

$$x' = -x + y, \quad y' = -y + u(t), \quad x(0) = y(0) = 0,$$

where $u(t) = 1$ for $0 \leq t < 1$ and $u(t) = 0$ for $t > 1$. Take the state to be continuous at the switch; use one-sided derivatives there. The value assigned to $u(1)$ does not change the continuous solution.

Tasks

1. Solve on $0 \leq t \leq 1$, then continue for $t \geq 1$ using $\tau = t - 1$ and the state at the switch.
2. Find the one-sided state derivatives at $t = 1$. Which component derivative is continuous, and which jumps?
3. Find the global forward maximum of x and its time. Explain why the maximum occurs after the input has switched off.
4. Find the long-time limit and the integrals $\int_0^\infty x(t) dt$ and $\int_0^\infty y(t) dt$ by integrating the differential equations.

Question 8 – Solution

Strategy. Match the state at the switch, then track the stored second-component response as it continues to feed the first.

Step 1: Solve and match the two intervals. For $0 \leq t \leq 1$, integration gives $x = 1 - (1+t)e^{-t}$, $y = 1 - e^{-t}$. Set $a = 1 - 2/e$, $b = 1 - 1/e$, the switch values. For $\tau = t - 1 \geq 0$, the homogeneous continuation is

$$x = e^{-\tau}(a + b\tau), \quad y = be^{-\tau}.$$

At $\tau = 0$ these match the first formulas, so both components are continuous.

Step 2: Compare one-sided derivatives. Before the switch, $(x', y') = (te^{-t}, e^{-t})$. Thus $X'(1^-) = (1/e, 1/e)^T$. Immediately after it, $X'(1^+) = (-a+b, -b)^T = (1/e, -1+1/e)^T$. The derivative x' is continuous; y' has jump -1 . A finite step input changes a derivative, not the state itself.

Step 3: Find the delayed first-component peak. Before switching, $x' > 0$ for $t > 0$. Afterward, $x' = e^{-\tau}(b - a - b\tau)$, with $b - a = 1/e$. It changes sign once, at $\tau = 1/(e - 1)$, hence

$$t_{\max} = 1 + \frac{1}{e - 1}, \quad x_{\max} = (1 - 1/e)e^{-1/(e-1)}.$$

The still-positive y initially exceeds x after the input ends, so $x' = -x + y$ remains positive until the two components become equal.

Step 4: Integrate the total response. Both components tend to zero exponentially times a linear factor, so their integrals converge. Integrating $y' = -y + u$ over $[0, \infty)$ and using its zero endpoints gives $\int y = \int u = 1$. Integrating $x' = -x + y$ similarly gives $\int_0^\infty x(t) dt = \int_0^\infty y(t) dt = 1$. Equal total areas do not mean equal graphs or simultaneous peaks. The figure marks the switch and the delayed maximum of x .

