

Question 6

For $R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$, consider

$$X' = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} X + \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

Investigate the response to a forcing that rotates at the homogeneous oscillation frequency.

Tasks

1. Use $X = R(t)Z$ to derive and solve a constant-forcing equation for Z . Give the full family.
2. Find the unique 2π -periodic solution and prove its uniqueness.
3. For $X(0) = 0$, find the state, its distance from the periodic solution at the same time, and its distance from the unit circle.
4. Does the zero-state response become exactly periodic at a finite time or tend to a single point? Explain how its state curve should be interpreted in this time-dependent forced problem.

Question 6 – Solution

Strategy. A rotating coordinate frame removes the oscillation and leaves scalar damping toward a constant vector.

Step 1: Pass to the rotating frame. Let $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, so $R' = JR$ and $A = -I + J$. Substitution cancels the rotational terms, giving $Z' = -Z + (1, 0)^T$. Hence

$$X(t) = R(t) \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{-t} \left(c - \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \right],$$

where $c = X(0)$ is an arbitrary real vector. This gives every solution.

Step 2: Identify and prove the periodic response. Taking $c = (1, 0)^T$ gives $P(t) = (\cos t, \sin t)^T$. If another solution were 2π -periodic, its difference from P would be $e^{-t}R(t)d$ and would satisfy $d = e^{-2\pi}d$ at times zero and 2π . Thus $d = 0$, proving uniqueness. Damping prevents a secular term despite the matched rotation frequency.

Step 3: Quantify zero-state convergence. For $c = 0$,

$$X(t) = (1 - e^{-t})(\cos t, \sin t)^T \quad (t \geq 0).$$

Since rotations preserve length, $\|X(t) - P(t)\| = e^{-t}$. Its radius is $1 - e^{-t}$, so its distance to the unit circle is also e^{-t} . Its initial velocity is $(1, 0)^T$, agreeing with the input at zero.

Step 4: Distinguish asymptotic response from exact return. The radius is strictly increasing at every finite forward time, so the solution neither repeats a state nor becomes exactly periodic later. Along $t = 2\pi n$ and $t = 2\pi n + \pi$, its states tend to $(1, 0)^T$ and $(-1, 0)^T$, respectively; there is no single point limit. The drawn curve is a nonautonomous state path approaching a periodic response, not an autonomous phase portrait or a claim of a finite-time merger with the circle. Only a finite initial segment is drawn.

Solid: initial state path; dashed: periodic-response circle

