

Question 5

Let

$$X' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} X + e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The forcing matches a repeated homogeneous exponent and enters through the second component of a length-two chain.

Tasks

1. Use $Z = e^t X$ to derive a simpler forced system and solve the IVP.
2. Test the trial $X_p = te^{-t}v$ with constant vector v . Explain why it cannot supply a particular solution here.
3. Find the forward maximum of each component and its time. Do the maxima occur together?
4. Determine the long-time behavior and compare the polynomial degree of this forced response with the homogeneous family. Does resonance necessarily imply an unbounded response?

Question 5 – Solution

Strategy. Removing the common exponential reveals successive integrations along the nilpotent chain.

Step 1: Integrate along the chain. With $N = A + I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, the transformed system is $Z' = NZ + (0, 1)^T$, $Z(0) = 0$. Thus $z_2' = 1$, $z_1' = z_2$, giving $z_2 = t$, $z_1 = t^2/2$ and

$$x = \frac{1}{2}t^2e^{-t}, \quad y = te^{-t}.$$

Substitution verifies the original equation and zero initial state.

Step 2: Reject an insufficient polynomial trial. The trial would give $Z = tv$, so its equation requires $v = tNv + (0, 1)^T$ for every time. Matching coefficients forces $Nv = 0$ and $v = (0, 1)^T$, but $N(0, 1)^T = (1, 0)^T \neq 0$. These conditions contradict each other. A factor t multiplying a constant vector is insufficient; the chain requires a quadratic term in the first component.

Step 3: Locate the separate response peaks. For $t \geq 0$, $x' = e^{-t}t(2-t)/2$ and $y' = e^{-t}(1-t)$. Their sign changes show

$$x_{\max} = 2e^{-2} \text{ at } t = 2, \quad y_{\max} = e^{-1} \text{ at } t = 1.$$

The maxima occur at different states and different times. The figure marks these component peaks on the two time graphs.

Step 4: Interpret resonance without growth to infinity. Both components tend to zero, since every fixed polynomial times e^{-t} vanishes. The homogeneous family is $e^{-t}(c_1 + c_2t, c_2)^T$, whose largest degree is one; the forcing raises the first component's degree to two. This is a resonant polynomial response, even though the negative exponential ultimately wins. Resonance describes an enhanced polynomial factor, not a universal guarantee of an unbounded solution. Adding any homogeneous solution still leaves forward decay.

