

Question 4

Consider the time-dependent system

$$X' = A(t)X + f(t), \quad A(t) = \begin{pmatrix} 0 & 2t \\ 0 & 0 \end{pmatrix}, \quad f(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Use variation of parameters and check the correct two-time transition matrix.

Tasks

1. Find a fundamental matrix $\Phi(t)$ with $\Phi(0) = I$ and its inverse.
2. Derive the variation-of-parameters equation by writing $X = \Phi C$, then compute the IVP solution.
3. Find $T(t, s) = \Phi(t)\Phi(s)^{-1}$ and express the same solution as an integral of $T(t, s)f(s)$. Evaluate it.
4. A proposed shortcut replaces $T(t, s)$ by $\Phi(t - s)$. Compute the resulting candidate and its residual in the original system. Explain why the shortcut fails.

Question 4 – Solution

Strategy. For time-dependent coefficients, propagation depends on both the departure time and the arrival time.

Step 1: Normalize the homogeneous matrix. Solving $y' = 0$, $x' = 2ty$ gives

$$\Phi(t) = \begin{pmatrix} 1 & t^2 \\ 0 & 1 \end{pmatrix}, \quad \Phi(t)^{-1} = \begin{pmatrix} 1 & -t^2 \\ 0 & 1 \end{pmatrix}.$$

Its determinant is one, and direct differentiation verifies $\Phi' = A(t)\Phi$ and $\Phi(0) = I$.

Step 2: Derive and integrate the varying coefficients. Substituting $X = \Phi C$ cancels $\Phi' C = A\Phi C$, leaving $C' = \Phi^{-1}f = (t - t^2, 1)^T$. Zero initial data gives $C = (t^2/2 - t^3/3, t)^T$. Therefore

$$X = (t^2/2 + 2t^3/3, t)^T.$$

Indeed $x' = t + 2t^2 = 2ty + t$ and $y' = 1$.

Step 3: Use the genuine two-time transition. Matrix multiplication yields

$$T(t, s) = \begin{pmatrix} 1 & t^2 - s^2 \\ 0 & 1 \end{pmatrix}.$$

Thus $X(t) = \int_0^t T(t, s)f(s) ds = \int_0^t (s + t^2 - s^2, 1)^T ds$, reproducing the same formula. This expression also follows directly from $\Phi(t) \int_0^t \Phi(s)^{-1}f(s) ds$.

Step 4: Expose the incorrect shortcut. Using $\Phi(t-s)$ instead gives $\tilde{X} = (t^2/2 + t^3/3, t)^T$. Its residual is

$$\tilde{X}' - A(t)\tilde{X} - f(t) = (-t^2, 0)^T,$$

which is not identically zero. The two matrices differ because $t^2 - s^2 \neq (t-s)^2$ in general. A fundamental matrix normalized at zero cannot automatically be shifted in time when the coefficient matrix itself changes with time.