

Question 3

For real constants a, b , consider

$$X' = \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} X + e^t \begin{pmatrix} a \\ b \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

A student says that because the forcing exponent equals an eigenvalue, a factor t must occur in a particular solution for every nonzero (a, b) .

Tasks

1. Determine exactly when a trial $X_p = e^t v$ with constant vector v works, and describe all such v .
2. Construct a particular solution valid for every (a, b) and then solve the zero-state IVP.
3. Explain how the component of the forcing in the resonant eigendirection decides whether a polynomial factor is unavoidable.
4. For $a = 0, b \neq 0$, determine the forward growth of the solution. Is growth by itself evidence of resonance? Address the student's claim.

Question 3 – Solution

Strategy. Resonance is a compatibility problem for the forcing direction, not merely a comparison of two scalar exponents.

Step 1: Test the simple exponential trial. Substitution gives $(I - A)v = (a, b)^T$, or $0 = a$, $3v_2 = b$. Thus the trial works exactly when $a = 0$, with $v = (c, b/3)^T$, $c \in \mathbb{R}$. The free first component is a homogeneous contribution, not a new forced effect.

Step 2: Construct the general forced response. A particular solution for all parameters is $X_p = (ate^t, (b/3)e^t)^T$. Adding $(c_1e^t, c_2e^{-2t})^T$ and imposing zero initial data gives

$$x = ate^t, \quad y = \frac{b}{3}(e^t - e^{-2t}).$$

Differentiation gives $x' - x = ae^t$ and $y' + 2y = be^t$, verifying both forcing components and the initial values.

Step 3: Identify the resonant component. The resonant eigendirection is $(1, 0)^T$. Its forced scalar equation is $(e^{-t}x)' = a$, so a nonzero a necessarily creates ate^t . When $a = 0$, there is no forcing in that direction and no polynomial factor is needed. No choice of homogeneous constants can cancel a nonzero ate^t for all time. The condition is directional, not just spectral.

Step 4: Separate forced growth from resonance. For $a = 0, b \neq 0$, the zero-state solution has $x = 0$ and $y/e^t \rightarrow b/3$. It grows because its forcing grows, although the forced second component has homogeneous rate -2 and is not resonant. The first eigenvalue 1 is present in the matrix but receives no input. Thus growth alone does not prove resonance, and the student's claim fails for every nonzero forcing vector of the form $(0, b)^T$.