

Question 2

Let

$$X' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} X + \begin{pmatrix} t \\ 1 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Both homogeneous eigenvalues are negative, but the forcing has an unbounded component. Use a polynomial particular solution.

Tasks

1. Find a particular solution of the form $X_p = ut + v$, with constant vectors u, v . Explain why a constant trial cannot suffice.
2. Add the complete homogeneous family and impose the initial data.
3. Verify the result in the original equations and determine the limits of $x(t) - t$ and $y(t)$.
4. Decide which components are bounded and whether two solutions with different initial data approach one another. Explain why homogeneous stability does not force convergence to a constant here.

Question 2 – Solution

Strategy. Match polynomial coefficients before imposing initial data; stability controls differences of responses, not the size of an unbounded input.

Step 1: Match the polynomial coefficients. Substituting $X_p = ut + v$ gives $u = A(ut + v) + (t, 1)^T$. The coefficient of t requires $Au = -(1, 0)^T$, hence $u = (1, 0)^T$. The constant equation $u = Av + (0, 1)^T$ gives $v = (0, 1)^T$. Thus $X_p = (t, 1)^T$. A constant trial cannot cancel the forcing's nonzero coefficient of t .

Step 2: Complete the family and solve the IVP. The homogeneous solution is $e^{-t}(c_1 + c_2t, c_2)^T$. Initial zero gives $c_1 = 0$, $c_2 = -1$, so

$$x = t(1 - e^{-t}), \quad y = 1 - e^{-t}.$$

Step 3: Check the original equation and asymptotics. We have $x' = 1 - e^{-t} + te^{-t}$ and $y' = e^{-t}$. The right sides $-x + y + t$ and $-y + 1$ give those same expressions. The initial values vanish. Moreover $x - t = -te^{-t} \rightarrow 0$ and $y \rightarrow 1$. The first component tracks a growing ramp; its tracking error decays, but its value does not tend to a finite constant.

Step 4: Separate stability from bounded response. On $t \geq 0$, y is bounded between zero and one, while $x \rightarrow \infty$. For any two solutions, their difference has the form $e^{-t}(d_1 + d_2t, d_2)^T \rightarrow 0$. Thus initial-data effects disappear even though the common forced response grows. Negative eigenvalues of A guarantee decay of the homogeneous transient; they do not turn an unbounded polynomial forcing into a bounded steady state. A complete answer therefore describes both the particular response and the transient.