

**Question 1**

Consider the constant-input system

$$X' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} X + \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \quad X(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Separate the equilibrium response from the transient response.

**Tasks**

1. Find the equilibrium and transform the equation into a homogeneous system by translating the state.
2. Use the homogeneous eigenmodes to find the complete family, then solve the stated IVP.
3. Determine whether either component overshoots its equilibrium value for  $t \geq 0$ . Give the limiting state and justify the signs of the derivatives.
4. Explain why the nonhomogeneous solution family is not a vector space. Which linear combinations of two solutions necessarily solve the same forced equation?

## Question 1 – Solution

**Strategy.** A constant particular solution shifts the equilibrium; the translated state evolves through the familiar homogeneous modes.

**Step 1: Translate to the forced equilibrium.** Solving  $-2x + y + 3 = 0$ ,  $x - 2y = 0$  gives  $X_* = (2, 1)^T$ . With  $Z = X - X_*$ , the equation becomes  $Z' = AZ$ . The origin is not an equilibrium of the forced system.

**Step 2: Resolve the transient coefficients.** The eigenpairs of  $A$  are  $-1, (1, 1)^T$  and  $-3, (1, -1)^T$ . Thus every solution is  $X_* + c_1 e^{-t}(1, 1)^T + c_2 e^{-3t}(1, -1)^T$ . The initial state requires  $c_1 = -3/2$ ,  $c_2 = -1/2$ , giving

$$x = 2 - \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t}, \quad y = 1 - \frac{3}{2}e^{-t} + \frac{1}{2}e^{-3t}.$$

**Step 3: Check monotonicity and the limit.** Here  $x' = \frac{3}{2}(e^{-t} + e^{-3t}) > 0$  for  $t \geq 0$ , while  $y' = \frac{3}{2}(e^{-t} - e^{-3t}) > 0$  for  $t > 0$  and  $y'(0) = 0$ . Both start at zero and tend to 2 and 1, respectively. Their monotone approach excludes overshoot; the initial velocity is  $(3, 0)^T$ , as the original equation requires. The two horizontal dotted lines in the figure are component limits, not additional time-dependent solutions.

**Step 4: Identify the affine superposition rule.** If  $U, V$  solve the same forced equation, then  $(\alpha U + \beta V)' - A(\alpha U + \beta V) = (\alpha + \beta)(3, 0)^T$ . It is a solution exactly when  $\alpha + \beta = 1$ . In particular the zero function is not a solution, and the sum of two solutions generally fails. Differences of solutions are homogeneous; adding any homogeneous solution to one particular solution gives the complete affine family derived above.

