

Question 9

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

You may design a second linear measurement to accompany $x + y = 1$:

$$ax + by = c.$$

It must select the target $(x, y) = (1/3, 2/3)$ uniquely. In a normalized design, require $a, b \geq 0$ and $a + b = 3$.

Tasks

1. Classify all real triples (a, b, c) that select the target uniquely. Explain the excluded case geometrically.
2. Under the normalization, describe every admissible design using one parameter and identify the excluded parameter value.
3. If the second right-hand side changes from c to $c + \delta$ while the first equation remains exact, derive the coordinate errors for an admissible design.
4. For $|\delta| \leq \eta$, where $\eta > 0$, find all normalized designs minimizing the worst possible absolute coordinate error. Give that minimum and verify the proposed designs directly.

Question 9 – Solution

Strategy. Passing through a target is necessary, but the measurement must also distinguish points along the first equation's line.

Step 1: Require both passage and uniqueness. The target lies on the second line exactly when $c = (a + 2b)/3$. Substituting $y = 1 - x$ gives $(a - b)x = c - b$. Hence unique selection requires

$$c = (a + 2b)/3, \quad a \neq b.$$

If $a = b$, the target condition forces $c = a$ and the second equation is a multiple of $x + y = 1$ (possibly $0 = 0$). The whole first line remains possible.

Step 2: List all normalized designs. Put $b = 3 - a$, with $0 \leq a \leq 3$. Then

$$b = 3 - a, \quad c = 2 - a/3, \quad a \in [0, 3] \setminus \{3/2\}.$$

The midpoint is excluded because both coefficients would equal $3/2$.

Step 3: Derive the exact measurement sensitivity. Subtract the target equations from the perturbed equations. With $\Delta x = x - 1/3$ and $\Delta y = y - 2/3$, one obtains $\Delta x + \Delta y = 0$ and $a\Delta x + b\Delta y = \delta$. Thus

$$\Delta x = \frac{\delta}{a - b}, \quad \Delta y = -\frac{\delta}{a - b}.$$

The worst coordinate error over $|\delta| \leq \eta$ is exactly $\eta/|a - b|$, attained at either endpoint of the allowed error interval.

Step 4: Optimize within the allowed normalization. Since $a + b = 3$ and both are nonnegative, $|a - b| \leq 3$, with equality exactly at $(a, b) = (3, 0)$ or $(0, 3)$. Therefore the minimum worst error is $\boxed{\eta/3}$, attained by exactly two designs:

$$3x = 1 \quad \text{or} \quad 3y = 2.$$

Perturbing these readings directly changes the measured coordinate by $\delta/3$ and the other by its negative. Both attain the claimed bound. Without the normalization, arbitrary rescaling would change the meaning of an absolute right-hand-side error, making this comparison ill-defined.