

Question 8

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

A steady flow network has directed internal flows p from A to B , q from B to C , and r from C to A . External flow enters A at rate 3; rates 1 and 2 leave B and C , respectively. There is no storage. All rates use the same units, and internal flows must be nonnegative.

Tasks

1. Write the three node-balance equations and identify any redundancy.
2. Find every nonnegative steady flow and explain why the external data do not determine the internal circulation uniquely.
3. Impose capacity bounds $p \leq 5$, $q \leq 4$ and $r \leq 3$. Find the entire feasible set and identify which capacities limit it.
4. Among feasible flows, minimize the cost rate $2p + q + 3r$. Prove uniqueness of the minimizer and interpret it in terms of circulation.

Question 8 – Solution

Strategy. Conservation fixes net transfers but can leave an arbitrary circulating flow around a closed loop.

Step 1: Balance every node. Inflow equals outflow, giving

$$A : r + 3 = p, \quad B : p = q + 1, \quad C : q = r + 2.$$

Equivalently $p - r = 3$, $q - p = -1$, $r - q = -2$. The three left sides sum to zero, as do their right sides. Any two balances imply the third because total external inflow equals total external outflow.

Step 2: Recover the free circulation. Set $r = t$. Then

$$(p, q, r) = (t + 3, t + 2, t), \quad t \geq 0.$$

The condition $t \geq 0$ is necessary from $r \geq 0$ and sufficient for all three flows to be nonnegative. Adding the same amount to all three internal flows preserves every node balance: it is an additional circulation around the loop.

Step 3: Apply all capacities. The bounds give $t \leq 2$, $t \leq 2$ and $t \leq 3$, respectively. Thus the entire feasible family has $[0 \leq t \leq 2]$. The capacities on p and q both limit the upper endpoint; at $t = 2$ they are saturated, while $r = 2 < 3$ leaves unused capacity on the third link.

Step 4: Minimize the cost on the feasible family. Substituting the flow formulas yields

$$2p + q + 3r = 2(t + 3) + (t + 2) + 3t = 8 + 6t.$$

This increases strictly on $[0, 2]$. Therefore the unique minimizer is $(p, q, r) = (3, 2, 0)$, with cost rate $[8]$. It sends the required external transfers without an added loop circulation. The diagram shows directions, not arrows scaled in proportion to flow.

