

Question 7

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

A sensor's response is assumed to be a quadratic polynomial $p(t) = a + bt + ct^2$. Exact readings at three times are

$$p(0) = 1, \quad p(1) = 2, \quad p(2) = 5.$$

Later, consider a perturbation of only the last reading to $5 + \epsilon$, where $|\epsilon| \leq \eta$ and $\eta > 0$.

Tasks

1. Find the unique polynomial from the exact readings using a system of equations. Predict $p(3)$.
2. Find the exact perturbed coefficients and the response error at an arbitrary time $T \geq 0$.
3. Identify all nonnegative times whose predicted value is insensitive to this perturbation. Find sharp uniform error bounds on $0 \leq T \leq 2$ and at $T = 10$.
4. Explain which conclusions depend on the quadratic model. Construct a nonquadratic smooth response with the same three original readings but a different value at 3.

Question 7 – Solution

Strategy. A model assumption converts function recovery into coefficient recovery, while extrapolation can amplify measurement errors.

Step 1: Solve the exact coefficient equations. The data give $a = 1$, $a + b + c = 2$ and $a + 2b + 4c = 5$. Thus $b + c = 1$ and $2b + 4c = 4$, yielding $c = 1$ and $b = 0$. Every coefficient is forced, so

$$p(t) = 1 + t^2, \quad p(3) = 10.$$

Step 2: Retain the measurement perturbation. The first two equations are unchanged, while $2b + 4c = 4 + \epsilon$. Solving gives

$$a = 1, \quad b = -\epsilon/2, \quad c = 1 + \epsilon/2, \quad p_\epsilon(T) - p(T) = \frac{\epsilon}{2}T(T-1).$$

The error vanishes at both unchanged measurement times and equals ϵ at $T = 2$, as required.

Step 3: Compare interpolation and extrapolation. The error is independent of ϵ exactly when $T(T-1) = 0$, so the nonnegative insensitive times are 0 and 1. On $[0, 1]$, $|T(T-1)| \leq 1/4$; on $[1, 2]$, $T(T-1)$ increases from 0 to 2. Therefore

$$\sup_{0 \leq T \leq 2} |p_\epsilon - p| \leq \eta, \quad |p_\epsilon(10) - p(10)| \leq 45\eta.$$

Both bounds are sharp at $|\epsilon| = \eta$, the first at $T = 2$. The figure shows the true curves for $\epsilon = 0, \pm 0.4$.

Step 4: State the model's limitation. For any real K , the smooth function $q_K(t) = 1 + t^2 + Kt(t-1)(t-2)$ matches all three original readings. For $K \neq 0$ it is not quadratic, and $q_K(3) = 10 + 6K \neq 10$. Thus three readings uniquely determine a member of the quadratic family, not an arbitrary smooth response. Extrapolation bounds also rely on the quadratic family and on the stated location of the measurement error.

