

Question 2

Work over the real numbers. Use substitution or elimination, keeping track of conditions under which an operation preserves all solutions. Check candidates in the original equations.

For real λ , solve

$$(\lambda - 1)x + y = 1, \quad x + (\lambda - 1)y = 1.$$

Tasks

1. Use the sum and difference of the equations to classify the solution set for every λ .
2. Explain why dividing by λ or $\lambda - 2$ before checking their zeros can lose entire cases. Verify the exceptional cases directly.
3. Find the limits of the unique solution as $\lambda \rightarrow 2$ and as $\lambda \rightarrow 0$ from either side. Compare these limits with the solution sets at the exceptional values.
4. Classify all λ for which a solution with $x, y \geq 0$ exists, giving the full nonnegative solution set whenever it is not a single point.

Question 2 – Solution

Strategy. Adding and subtracting separates the sum of the unknowns from their difference, exposing both exceptional parameter values.

Step 1: Reduce reversibly. The sum and difference equations are

$$\lambda(x + y) = 2, \quad (\lambda - 2)(x - y) = 0.$$

They are equivalent to the originals: taking half their sum and half their difference recovers the two original equations. If $\lambda \neq 0, 2$, then

$$x = y = 1/\lambda.$$

At $\lambda = 0$, the sum is $0 = 2$, giving no solution. At $\lambda = 2$, only $x + y = 1$ remains, giving infinitely many solutions.

Step 2: Retain the exceptional equations. Directly, $\lambda = 0$ gives $-x + y = 1$ and $x - y = 1$, incompatible parallel lines. At $\lambda = 2$, both equations become $x + y = 1$. Dividing by a factor that can be zero discards these cases rather than solving them. The figure displays the two exceptional geometries.

Step 3: Compare limits with exact parameter values. As $\lambda \rightarrow 2$ through the unique-solution cases, $(x, y) \rightarrow (1/2, 1/2)$. This is one point of the entire line present at $\lambda = 2$, not the whole limiting parameter's solution set. As $\lambda \rightarrow 0^+$ both coordinates increase without bound; as $\lambda \rightarrow 0^-$ both decrease without bound. There is no finite limiting solution, consistent with the contradiction at zero.

Step 4: Enforce nonnegativity. For $\lambda \neq 0, 2$, the solution is nonnegative exactly when $\lambda > 0$. At $\lambda = 2$, the nonnegative solutions are $(x, y) = (u, 1 - u)$, $0 \leq u \leq 1$. Therefore a nonnegative solution exists exactly for $\lambda > 0$, with the entire stated segment at 2.

