

Question 10

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let $f \in C^1[0, \infty)$ satisfy $|f'(t)| \leq L$, with $L \geq 0$. For $\varepsilon > 0$ define

$$k_\varepsilon(t) = \frac{1}{\varepsilon} e^{-t/\varepsilon}, \quad y_\varepsilon = k_\varepsilon * f.$$

This causal averaging kernel becomes concentrated near zero as ε decreases.

Tasks

1. Derive an IVP for y_ε and use integration by parts to express $y_\varepsilon - f$ in terms of $f(0)$ and f' .
2. Prove an explicit error bound containing both the initial transient and a term at most $L\varepsilon$.
3. Distinguish uniform convergence on $[0, \infty)$ from uniform convergence on $[t_0, \infty)$ for $t_0 > 0$. Show the $L\varepsilon$ bound is sharp when $f(0) = 0$.
4. For $f(t) = 1 + t$, decide separately whether $\varepsilon = 1/20$ and $\varepsilon = 1/40$ ensure $|y_\varepsilon(t) - f(t)| \leq 1/20$ for every $t \geq 1$. Justify the strict inequalities involved.

Question 10 – Solution

Strategy. Separate the error due to starting from rest from the error due to variation of the input.

Step 1: Obtain the equation and exact error identity. Differentiation gives $\varepsilon y'_\varepsilon + y_\varepsilon = f$, $y_\varepsilon(0) = 0$. Integrating the convolution by parts in u gives

$$y_\varepsilon(t) - f(t) = -e^{-t/\varepsilon} f(0) - \int_0^t e^{-(t-u)/\varepsilon} f'(u) du.$$

Step 2: Bound the two contributions. Taking absolute values yields

$$|y_\varepsilon - f| \leq |f(0)|e^{-t/\varepsilon} + L\varepsilon(1 - e^{-t/\varepsilon}).$$

This accounts for the kernel mass missing before time zero as well as the variation over its recent history.

Step 3: Distinguish the convergence statements. If $f(0) = 0$, the error is uniformly at most $L\varepsilon$ on the half-line. For the ramp $f = Lt$ it equals $L\varepsilon(1 - e^{-t/\varepsilon})$ in magnitude, whose supremum is $L\varepsilon$ (approached as $t \rightarrow \infty$ when $L > 0$). Thus the constant is sharp. If $f(0) \neq 0$, the error at zero is always $|f(0)|$, preventing uniform convergence on $[0, \infty)$. On $[t_0, \infty)$ it is bounded by $|f(0)|e^{-t_0/\varepsilon} + L\varepsilon \rightarrow 0$ for every fixed $t_0 > 0$.

Step 4: Test the two proposed averaging scales. For $f = 1 + t$, $y_\varepsilon = t + 1 - \varepsilon - (1 - \varepsilon)e^{-t/\varepsilon}$. For $0 < \varepsilon < 1$ the error magnitude is $E_\varepsilon = \varepsilon + (1 - \varepsilon)e^{-t/\varepsilon}$. At $\varepsilon = 1/20$ this is strictly greater than $1/20$ at every finite time, so that choice fails. At $\varepsilon = 1/40$ and $t \geq 1$,

$$E_\varepsilon \leq \frac{1}{40} + \frac{39}{40}e^{-40} < \frac{1}{20},$$

since $e^{40} > 1 + 40 = 41 > 39$. Thus the smaller choice succeeds. The figure shows the initial error layer and the $1/20$ tolerance.

