

Question 9

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Two normalized stages $k_\alpha(t) = \alpha e^{-\alpha t}$ and $k_\beta(t) = \beta e^{-\beta t}$, with $\alpha, \beta > 0$, are followed by a delay $d \geq 0$:

$$r(t) = H_d(t)(k_\alpha * k_\beta)(t-d).$$

The observed response is positive precisely for $t > 1$, has mass 1, mean $\mu = 5/2$ and variance $v = 5/4$. For a nonnegative unit-mass kernel p , $\mu_p = \int_0^\infty tp(t) dt$ and $v_p = \int_0^\infty (t - \mu_p)^2 p(t) dt$.

Tasks

1. Prove that convolution adds the means and variances of two nonnegative unit-mass kernels with finite second moments.
2. Compute the mean and variance of k_α , and explain the effect of a delay on mass, mean and variance.
3. Recover d and the two rates from the observations. State precisely what is and is not uniquely identifiable.
4. For general prescribed delay d , mean μ and variance v , give necessary and sufficient conditions for two finite positive rates to exist. Interpret the equality case and the excluded upper boundary.

Question 9 – Solution

Strategy. Work with reciprocal rates. Convolution moment identities reduce identification to a quadratic equation.

Step 1: Prove the moment identities. In the convolution integral use $t = u + w$, $u, w \geq 0$. Nonnegativity permits interchange, and finite second moments make the following values finite:

$$\begin{aligned}\int (p * q) &= \left(\int p\right)\left(\int q\right) = 1, \\ \mu_{p*q} &= \iint (u + w)p(u)q(w) du dw = \mu_p + \mu_q, \\ \int t^2(p * q)(t) dt &= \int u^2 p(u) du + 2\mu_p \mu_q + \int w^2 q(w) dw.\end{aligned}$$

Subtracting $(\mu_p + \mu_q)^2$ from the last line proves $\boxed{v_{p*q} = v_p + v_q}$.

Step 2: Calculate the stage statistics. Integration by parts gives mass 1, first moment $1/\alpha$ and second moment $2/\alpha^2$ for k_α . Its variance is therefore $1/\alpha^2$. A delay replaces time by $t = d + u$: mass stays 1, the mean increases by d , and the centered second moment is unchanged. Hence

$$\mu = d + \frac{1}{\alpha} + \frac{1}{\beta}, \quad v = \frac{1}{\alpha^2} + \frac{1}{\beta^2}.$$

Step 3: Recover the parameters. Both stage kernels are positive for positive times, so their convolution is positive at every positive time and zero at zero. The observed onset thus forces $d = 1$. Put $x = 1/\alpha$, $z = 1/\beta$. Then

$$x + z = \frac{3}{2}, \quad x^2 + z^2 = \frac{5}{4}, \quad xz = \frac{1}{2}, \quad (X - x)(X - z) = X^2 - \frac{3}{2}X + \frac{1}{2}.$$

Its roots are 1 and $1/2$, giving $\boxed{\{\alpha, \beta\} = \{1, 2\}}$. The delay and unordered rate pair are unique. Stage order cannot be identified even from the entire output because convolution is commutative.

Step 4: Give the full feasibility condition. Write $m = \mu - d$. Positive x, z require and are possible exactly when

$$\boxed{m > 0, \quad \frac{m^2}{2} \leq v < m^2.}$$

Indeed $xz = (m^2 - v)/2 > 0$ and $(x - z)^2 = 2v - m^2 \geq 0$; conversely these conditions make $(m \pm \sqrt{2v - m^2})/2$ real and strictly positive. At the lower equality the rates coincide. At the upper boundary one reciprocal rate would be zero, requiring an infinite rate, excluded here.