

Question 8

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Suppose an ordinary continuous input f produces $z = e^{-t} * f$. No impulses and no nonzero initial state are allowed. Consider the desired output $z_0(t) = te^{-t}$ and the perturbed outputs

$$z_{\varepsilon,N}(t) = te^{-t} + \varepsilon e^{-t} \sin(Nt), \quad \varepsilon > 0, \quad N \in \{1, 2, \dots\}.$$

Tasks

1. Characterize all continuously differentiable outputs obtainable from such inputs, and prove uniqueness of the recovered input.
2. Recover the input for z_0 . Decide whether the target $z(t) = e^{-t}$ is obtainable, explaining why differentiating alone can give a misleading answer.
3. Recover the input for $z_{\varepsilon,N}$ and bound its output error in the uniform norm on $[0, \infty)$. Compute the input error at zero.
4. Choose a sequence $\varepsilon = \varepsilon_N \rightarrow 0$ for which the output errors tend uniformly to zero but the input errors grow without bound. Interpret what this means for reconstruction from approximate measurements.

Question 8 – Solution

Strategy. Inversion differentiates the output. Check the initial-value restriction before using that derivative formula.

Step 1: Derive and prove the inverse rule. Differentiation under the integral gives $z' = f - z$, $z(0) = 0$. Therefore a C^1 target is obtainable exactly when it starts at zero, and its only possible continuous input is $\boxed{f = z' + z}$. Conversely, for this input, the target and the convolution solve the same IVP $w' + w = f$, $w(0) = 0$. Uniqueness proves they agree.

Step 2: Check two targets. For $z_0 = te^{-t}$, $z'_0 + z_0 = e^{-t}$, so the input is $f_0 = e^{-t}$. The target e^{-t} is impossible because its value at zero is 1. Although its derivative plus itself is zero, a zero input with zero initial state produces the zero output. Differentiation discards the missing initial condition.

Step 3: Differentiate the perturbation. The two exponential-derivative terms cancel, leaving

$$\boxed{f_{\varepsilon,N} = e^{-t} + \varepsilon N e^{-t} \cos(Nt)}.$$

All these inputs are continuous, and the target outputs vanish at zero. The uniform output error is at most ε because $e^{-t}|\sin Nt| \leq 1$. The input error at zero is εN ; in fact this is its uniform norm since $e^{-t}|\cos Nt| \leq 1$.

Step 4: Exhibit the instability. Take $\varepsilon_N = N^{-1/2}$. Then

$$\|z_{\varepsilon_N,N} - z_0\|_{\infty} \leq N^{-1/2} \rightarrow 0, \quad \|f_{\varepsilon_N,N} - f_0\|_{\infty} = \sqrt{N} \rightarrow \infty.$$

Thus the inverse is not continuous in the uniform output norm, even on smooth attainable outputs. Small high-frequency measurement errors can produce large reconstruction errors. The figure uses $\varepsilon = 0.05$, $N = 12$; each panel compares its own exact and perturbed functions.

