

Question 7

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let the unstable kernel be $h(t) = e^t$. First allow a piecewise continuous input f supported in $[0, T]$, where $T > 0$. Then specialize to

$$f_c(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -c, & 1 \leq t < 2, \\ 0, & t \geq 2, \end{cases} \quad c > 0,$$

with output $y = h * f_c$.

Tasks

1. For the general input, derive a necessary and sufficient integral condition for the output to vanish for every $t \geq T$.
2. For f_c , compare the choice that makes the ordinary input area zero with the choice that eliminates the output tail.
3. For the tail-eliminating choice, derive the complete piecewise output and find its maximum.
4. Find the transformed output for general c and compare its exact real transform domain before and after cancellation. Explain why the common transform domain of the factors need not be the output's full domain.

Question 7 – Solution

Strategy. An unstable kernel remembers a weighted input area. Equal positive and negative ordinary areas need not cancel that memory.

Step 1: Factor the growing exponential. For an input supported in $[0, T]$,

$$(h * f)(t) = e^t \int_0^{\min(t, T)} e^{-u} f(u) du.$$

Thus the output is zero for every $t \geq T$ if and only if $\int_0^T e^{-u} f(u) du = 0$, since e^t never vanishes.

Step 2: Distinguish the two cancellations. The ordinary area is $1 - c$, so it vanishes at $c = 1$. The weighted area is $(1 - e^{-1}) - c(e^{-1} - e^{-2})$; its unique zero is $c = e$. At $c = 1$ it equals $(1 - e^{-1})^2 > 0$, leaving a growing tail.

Step 3: Recover the selected response. Using the partial weighted integrals with $c = e$ gives

$$y_*(t) = \begin{cases} e^t - 1, & 0 \leq t < 1, \\ e - e^{t-1}, & 1 \leq t < 2, \\ 0, & t \geq 2. \end{cases}$$

The pieces agree at both junctions. It increases up to 1 and decreases from 1 to 2, so its unique maximum is $e - 1$ at $t = 1$.

Step 4: Track the transform domain. The convolution theorem initially gives, for $s > 1$,

$$Y(s) = \frac{1 - (1 + c)e^{-s} + ce^{-2s}}{s(s - 1)}.$$

If $c \neq e$, the output has a nonzero e^t tail and exact real domain $s > 1$. If $c = e$, the output is compactly supported, so its transform exists for every real s ; both apparent singularities at 0 and 1 are removable. The product formula agrees with it on $s > 1$ and extends by removal of these singularities. Cancellation changes this output, not the unstable kernel. The dashed comparison uses $c = 1$, which cancels only the ordinary input area.

