

Question 5

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

For a real parameter λ , solve the Volterra integral equation

$$u(t) = 1 + \lambda \int_0^t e^{-(t-\tau)} u(\tau) d\tau, \quad t \geq 0.$$

Seek a continuous solution on the entire half-line.

Tasks

1. Use transforms to derive the solution, treating every exceptional parameter value explicitly.
2. Derive an equivalent first-order IVP and prove the converse implication, establishing uniqueness among continuous solutions.
3. Classify boundedness, positivity and long-time behavior for all real λ , including $\lambda = 0$.
4. Give the exact real transform domain in every case. Explain why exponential decay of the memory kernel does not by itself guarantee a bounded solution.

Question 5 – Solution

Strategy. Feedback changes the effective growth rate. Verify the transformed candidate through an equivalent IVP so no growth assumption is needed for uniqueness.

Step 1: Solve the transformed equation. Initially for sufficiently large s ,

$$U = \frac{1}{s} + \frac{\lambda U}{s+1}, \quad U = \frac{s+1}{s(s+1-\lambda)}.$$

Partial fractions give

$$u(t) = \frac{1 - \lambda e^{(\lambda-1)t}}{1 - \lambda} \quad (\lambda \neq 1), \quad u(t) = 1 + t \quad (\lambda = 1).$$

In particular $\lambda = 0$ gives $u = 1$; the factor $s + 1$ then cancels in U .

Step 2: Prove equivalence without assuming a transform. For continuous u , define $v = e^{-t} * u$. Differentiation gives $v' = u - v$, $v(0) = 0$. The integral equation makes u continuously differentiable and yields

$$u' = (\lambda - 1)u + 1, \quad u(0) = 1.$$

Conversely, let u solve this IVP and form v as above. The residual $w = u - 1 - \lambda v$ satisfies $w' = -u + 1 + \lambda v = -w$ and $w(0) = 0$. Hence $w = 0$ and the integral equation holds, including at $\lambda = 0$. The linear IVP has a unique global solution, which proves the asserted existence and uniqueness.

Step 3: Classify all parameters. For $\lambda < 1$, the solution is bounded and tends to $1/(1 - \lambda) > 0$. For $\lambda = 1$ it grows linearly; for $\lambda > 1$ it grows exponentially with positive leading coefficient $\lambda/(\lambda - 1)$. For $\lambda \neq 1$, $u' = \lambda e^{(\lambda-1)t}$: it decreases from 1 to a positive limit when $\lambda < 0$, is constant when $\lambda = 0$, and increases when $\lambda > 0$. Thus it is positive for every real λ and every $t \geq 0$.

Step 4: Inspect the actual transform domain. For $\lambda < 1$ the positive nonzero limiting constant forces $s > 0$; for $\lambda = 1$ the linear term also requires $s > 0$. For $\lambda > 1$ the nonzero exponential requires $s > \lambda - 1$. Consequently

$$s > 0 \quad (\lambda \leq 1), \quad s > \lambda - 1 \quad (\lambda > 1).$$

Positive feedback can offset or exceed the kernel's decay. The rate in the equivalent equation is $\lambda - 1$, not simply the kernel's rate -1 .