

Question 4

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

Let f be real and piecewise continuous on each finite interval, with $|f(t)| \leq 1$. Consider

$$y'' + 2y' + 2y = f(t), \quad y(0) = y'(0) = 0.$$

We seek the smallest constant M such that $|y(t)| \leq M$ for every admissible input and every $t \geq 0$.

Tasks

1. Find the impulse-response kernel h and express y by convolution. Compute $\int_0^\infty h$.
2. Compute $\int_0^\infty |h|$ exactly by summing half-periods, and prove it is a uniform response bound.
3. Construct an input attaining $\int_0^T |h|$ at any prescribed $T > 0$. Prove that your uniform bound is sharp, and decide whether it can be attained at a finite time.
4. Give an explicit two-level input on $[0, 2\pi]$ whose response at 2π exceeds $1/2$. Explain the error in using the signed kernel integral as the universal bound.

Question 4 – Solution

Strategy. A sign-changing kernel can reinforce a suitably chosen sign-changing input. Its signed area loses that information.

Step 1: Recover the kernel. The zero-data transform gives $Y = F/[(s+1)^2 + 1]$, so

$$h(t) = e^{-t} \sin t, \quad y = h * f, \quad \int_0^\infty h(t) dt = \frac{1}{2}.$$

For example, the last integral follows from the antiderivative $-e^{-t}(\sin t + \cos t)/2$.

Step 2: Sum the absolute areas. Put $q = e^{-\pi}$. The first positive lobe has area $A_0 = (1+q)/2$. Each later absolute lobe is q times its predecessor, so

$$M = \int_0^\infty |h| = \frac{1+q}{2(1-q)}.$$

For every admissible input, $|y(t)| \leq \int_0^t |h(t-u)| du \leq M$.

Step 3: Prove sharpness and distinguish attainment. For any $T > 0$, set

$$f_T(u) = \operatorname{sgn}(\sin(T-u)) \quad (0 \leq u \leq T), \quad f_T(u) = 0 \quad (u > T),$$

with arbitrary values in $[-1, 1]$ at zeros. This input is admissible and $y(T) = \int_0^T |h(T-u)| du = \int_0^T |h(v)| dv$. These values approach M as $T \rightarrow \infty$, so any smaller proposed constant fails for some finite T . However, the absolute kernel has a strictly positive tail after every finite T , so $|y(T)| < M$ for every input at every finite time. The universal supremum is sharp but is not attained.

Step 4: Refute the signed-area bound explicitly. Take $f = -1$ on $(0, \pi)$, $f = 1$ on $(\pi, 2\pi)$ and zero afterward. This is the preceding construction for $T = 2\pi$. Therefore

$$y(2\pi) = A_0(1+q) = \frac{(1+q)^2}{2} > \frac{1}{2}.$$

The signed integral describes cancellation within the kernel, as for a constant input's limiting response. An arbitrary bounded input can reverse its own sign so that both lobes contribute positively; the absolute area is the appropriate worst-case quantity.