

Question 3

All functions are causal (zero for $t < 0$). Use the one-sided Laplace transform and $(f * g)(t) = \int_0^t f(t-u)g(u) du$. Write $H_a(t) = 0$ for $t < a$ and 1 for $t \geq a$. Values at isolated endpoints do not affect an ordinary integral.

An undamped oscillator receives a rapidly vanishing force:

$$y'' + y = e^{-t^2}, \quad y(0) = 2, \quad y'(0) = -1.$$

You may leave Gaussian integrals unevaluated and use $\int_0^\infty e^{-u^2} du = \sqrt{\pi}/2 < 1$.

Tasks

1. Find $Y(s)$ in terms of $G(s) = \mathcal{L}\{e^{-t^2}\}$ and write a convolution formula for y .
2. Verify the equation and both initial conditions by differentiating your convolution.
3. Find a limiting sinusoidal expression $y_\infty(t)$ with constant coefficients and prove $|y - y_\infty| \leq e^{-t^2}/(2t)$ for $t > 0$.
4. Prove that the response does not tend to zero, despite the vanishing force. Identify which feature of the homogeneous equation makes this possible.

Question 3 – Solution

Strategy. Keep the Gaussian force inside an integral and separate the permanent oscillation from the remaining tail integral.

Step 1: Transform and invert. The initial terms give $(s^2 + 1)Y - 2s + 1 = G$, hence

$$Y = \frac{2s - 1 + G}{s^2 + 1}, \quad \boxed{y = 2 \cos t - \sin t + z(t), \quad z(t) = \int_0^t \sin(t - u)e^{-u^2} du.}$$

These transform operations are valid for real $s > 0$.

Step 2: Differentiate with the endpoint terms. The sine kernel vanishes at its endpoint, so $z' = \int_0^t \cos(t - u)e^{-u^2} du$. Differentiating again gives $z'' = e^{-t^2} - z$ because $\cos 0 = 1$. Also $z(0) = z'(0) = 0$. Thus the formula satisfies the full IVP.

Step 3: Isolate the asymptotic oscillation. Set $A = \int_0^\infty e^{-u^2} \cos u du$ and $B = \int_0^\infty e^{-u^2} \sin u du$, both absolutely convergent. Expanding $\sin(t - u)$ gives

$$y_\infty = (2 - B) \cos t + (A - 1) \sin t, \quad y - y_\infty = - \int_t^\infty \sin(t - u)e^{-u^2} du.$$

For $t > 0$, use $1 \leq u/t$ on the tail:

$$|y - y_\infty| \leq \int_t^\infty e^{-u^2} du \leq \frac{1}{t} \int_t^\infty ue^{-u^2} du = \boxed{\frac{e^{-t^2}}{2t}}.$$

Step 4: Establish persistent motion. Since $|A| \leq \sqrt{\pi}/2 < 1$, the coefficient $A - 1$ is nonzero. The limiting sinusoid therefore has positive amplitude and has recurring positive and negative extrema. Along their times the error tends to zero, so y cannot tend to zero (or to any single constant). The homogeneous modes $\cos t, \sin t$ have no damping. The figure compares the exact integral solution with y_∞ ; the two quickly become indistinguishable.

